Quantitative techniques

The material contained in this chapter has been compiled to provide you with the mathematical background you will need to answer questions in your IB Economics exams involving the use of diagrams and graphs, as well as the use of mathematical techniques.

Sections 1 and 2 of this chapter are intended for students studying economics at both standard and higher levels. Section 3 is intended for students taking the course at higher level only, and focuses on the quantitative methods you will need to answer questions in higher level paper 3.

Students sometimes come to the study of economics concerned that the course may be based on mathematical techniques they might be unable to handle. However, as you will discover in the pages that follow, the mathematics you need to do well in your IB course is quite limited. The few simple techniques you will find here repeat themselves in a variety of applications in your IB course. Most if not all of these techniques involve mathematical ideas or methods you have very likely already encountered in your earlier years as a student or as part of your IB studies. Therefore, most of what is included in this chapter will be simply a review for you.

Note that you will find answers to all the 'Test your understanding' questions in this chapter on the teacher support website at ibdiploma.cambridge.org.

1 Percentages and percentage changes

Percentages

Why percentages are important

'Per cent' means 'out of 100'. It is simply another way of expressing a fraction or a decimal (which is also a kind of fraction) as a number in relation to 100.

Percentages allow us easily to make comparisons between fractions that cannot otherwise be easily compared. Suppose you take two tests, and you score $\frac{17}{20}$ in one and $\frac{29}{35}$ in the other. In which test did you get the higher score? There are only two ways to answer this question. One is to the find the lowest common denominator of the fractions, which is 140, and express the first score as $\frac{119}{140}$ and the second as $\frac{116}{140}$. This shows you scored better in the first test. A much simpler way, however, is to express the two scores as percentages, which are 85% for the first and 82.8% for the second. The use of percentages is not only a much easier method, but it also allows you to compare all your other test scores with each other.

Percentages in relation to fractions and decimals

Changing a fraction and a decimal into a percentage

Let's say we would like to express a fraction, such as $\frac{34}{75}$, as a percentage:

(i) We divide the numerator by the denominator, which converts the fraction into a decimal.

(ii) We multiply by 100%:

$$\frac{34}{75} = 0.4533; \qquad 0.4533 \times 100\% = 45.33\%$$

Changing a percentage into a fraction

To change a percentage into a fraction, we do the following:

(i) divide the percentage by 100 and remove the % sign

(ii) simplify the resulting fraction.

For example:

$$50\% = \frac{50}{100} = \frac{1}{2} \qquad 78\% = \frac{78}{100} = \frac{39}{50}$$

$$175\% = \frac{175}{100} = \frac{7}{4}$$

Changing a percentage into a decimal

We divide the percentage by 100 (move the decimal point two places to the left) and remove the % sign:

$$50\% = 0.50 \qquad 78\% = 0.78 \qquad 175\% = 1.75$$

Using percentages: examples

Example 1: Convert the test scores noted earlier, $\frac{17}{20}$ and $\frac{29}{35}$, into percentages.

$$\frac{17}{20} \times 100\% = 0.850 \times 100\% = 85.0\%$$

$$\frac{29}{35} \times 100\% = 0.829 \times 100\% = 82.9\%$$

(Note that when we multiply by 100%, we simply move the decimal point two places to the right and add the % sign.)

Example 2: In 2009, the total population of a country called Mountainland was 46.3 million. The rural population (living outside of cities) of Mountainland in the same year was 17.7 million. What percentage of Mountainland's population was rural?

We express the rural population as a fraction of the total population, and convert this fraction into a percentage:

$$\text{\% of population that is rural} = \frac{17.7 \textit{ million}}{46.3 \textit{ million}} \times 100\%$$

$$= 0.3823 \times 100\% = 38.23\%$$

Example 3: In a country called Flatland, 22 million people live in cities and 38 million live in the countryside. What percentage of Flatland's population lives in cities?

The total population is:

$$22 \text{ million} + 38 \text{ million} = 60 \text{ million}$$

Therefore, the percentage of people in cities is:

$$\frac{22}{60} \times 100\% = 36.7\%$$

Example 4: 12% of a graduating class of 150 students plan to study economics at university. How many students plan to study economics?

When we want to find a number that corresponds to a percentage of another given number, we multiply the given number by the percentage in decimal form:

$$12\% = 0.12; \qquad 0.12 \times 150 = 18 \text{ students}$$

Example 5: 70% of 140 students are actively involved in sports. How many students are involved in sports?

$$70\% = 0.70; \qquad 0.70 \times 140 = 98 \text{ students}$$

Test your understanding 1

1 Change the following decimals into percentages:
(a) 0.573 **(b)** 0.628 **(c)** 1.247 **(d)** 0.645.

2 Change the following fractions into percentages:
(a) $\frac{3}{9}$ **(b)** $\frac{271}{977}$ **(c)** $\frac{175}{65}$ **(d)** $\frac{5}{178}$.

3 Change the following percentages into decimals:
(a) 24.5% **(b)** 25% **(c)** 99% **(d)** 125%.

4 In a class of 25 students, 15 came to school by bus, and the rest walked to school. **(a)** What percentage of students came by bus? **(b)** What percentage of students walked?

5 A business makes a profit of 25% of its sales of \$17 000. How much profit does it make?

6 In 2009, 14% of Mountainland's population (of 46.3 million) were university graduates. How many people does this correspond to?

Percentage changes

Students studying economics at standard level should be able to understand and interpret percentage changes, but will not have to perform calculations in exams. Students taking the course at higher level should be able to calculate percentage changes.

Given two numbers, how to calculate a percentage change

To calculate a percentage change, we must have an initial number and a final number. The percentage change expresses the change (an increase or a decrease) as a percentage of the initial number. Suppose we want to find the percentage change from 50 to 75. We express the change as a fraction of the initial number and then convert this into a percentage. The change is 25 (=75−50), and the initial number is 50. Therefore, we have $\frac{25}{50} = 0.50$, which is equivalent to 50%.

More generally, we can calculate a percentage change in a variable, A, by using the following formula:

$$\text{\% change in A} = \frac{\textit{final value of A} - \textit{initial value of A}}{\textit{initial value of A}} \times 100\%$$

A *percentage increase* or *percentage decrease* is shown by whether the percentage change that is calculated by use of this formula is a positive or negative number.

Example 6: Suppose the population of Mountainland was 45.7 million in 2008 and 46.3 million in 2009. What was the percentage change in population from 2008 to 2009?

Applying the formula, we have

% change in population

$$= \frac{46.3\ million - 45.7\ million}{45.7\ million} \times 100 = \frac{0.6\ million}{45.7\ million} \times 100\%$$
$$= 0.013 \times 100\% = 1.3\%$$

The population of Mountainland thus *increased* by 1.3% (1.3 is a positive number). 1.3% is a *percentage increase*.

Example 7: Now suppose that the rural population of Mountainland was 18.3 million in 2008 and 17.7 million in 2009. What was the percentage change?

% change in rural population

$$= \frac{17.7\ million - 18.3\ million}{18.3\ million} \times 100\% = \frac{-0.6\ million}{18.3\ million} \times 100\%$$
$$= -0.033 \times 100\% = -3.3\%$$

The rural population of Mountainland therefore *decreased* by 3.3% (–3.3 is a negative number).

Example 8: The data in the table below show Mountainland's real GDP (real output produced) for the period 2008–10. Calculate *the rate of growth* in real GDP in (a) 2008–9, and (b) 2009–10.

Year	Real GDP (in trillion Mnl, the national currency)
2008	5.6
2009	5.7
2010	5.5

A percentage change may sometimes be referred to as 'rate of growth', which may be positive or negative.

(a) Rate of growth in real GDP, 2008–9:

% growth in real GDP (2008–2009)

$$= \frac{5.7\ trillion - 5.6\ trillion}{5.6\ trillion} \times 100\% = \frac{0.1\ trillion}{5.6\ trillion} \times 100\%$$
$$= 0.018 \times 100\% = 1.8\%$$

Mountainland experienced a positive rate of growth in real GDP of 8% in 2008–9.

(b) Rate of growth in real GDP, 2009–10:

% growth in real GDP (2009–2010)

$$= \frac{5.5\ trillion - 5.7\ trillion}{5.7\ trillion} \times 100\% = \frac{-0.2\ trillion}{5.7\ trillion} \times 100\%$$
$$= -0.035 \times 100\% = -3.5\%$$

Mountainland experienced a negative rate of growth in real GDP of –3.5% in 2009–10.

In practice, when multiplying a decimal by 100% to convert it into a percentage, we drop the '%' so it looks like we are multiplying the decimal by '100'.

Given a number and its percentage change, how to calculate the change and the final number

Example 9: Due to a combination of a higher birth rate and a large influx of immigrants into Mountainland, the population increased by 4.8% in the period 2009–10. The size of the population was 46.3 million in 2009. (a) How many people were added to Mountainland's population in 2009–10? (b) What was the size of the population in 2010 as a result of the 4.8% increase?

(a) The number of people added is 4.8% of the whole population in 2009 (46.3 million). Therefore:

$$4.8\% \times 46.3 \text{ million} = 0.048 \times 46.3 \text{ million}$$
$$= 2.2 \text{ million people}$$

(b) We can find the 2010 population in two ways:

(i) Add the increase of 2.2 million to the initial population of 46.3 million:

$$2.2 \text{ million} + 46.3 \text{ million} = 48.5 \text{ million in 2010}$$

(ii) A more direct way is to begin with the initial population of 46.3 million, and multiply this by 1 plus the percentage change in decimal form, or 1 + 0.048 = 1.048:

$$\text{population in 2010} = 46.3 \text{ million} \times 1.048$$
$$= 48.5 \text{ million people in 2010}$$

Note that the answers obtained in the two ways are identical.

Example 10: The rural population in Mountainland, which was 17.7 million in 2009, fell by 3.8% in 2009–10. What was the size of the rural population in 2010?

Again, we can do this in two ways:

(i) Find by how much the rural population fell, and subtract this from the rural population in 2009:

$$\text{rural population decrease 2009–10} = 3.8\% \times 17.7 \text{ million} = 0.038 \times 17.7 \text{ million} = 0.7 \text{ million people}$$

$$\text{rural population in 2010} = 17.7 \text{ million} - 0.7 \text{ million}$$
$$= 17.0 \text{ million people}$$

(ii) In a more direct way, we take the initial population of 17.7 million, and multiply this by 1 minus the percentage change in decimal form, or 1 – 0.038 = 0.962:

$$\text{rural population in 2010} = 17.7 \text{ million} \times 0.962$$
$$= 17.0 \text{ million people}$$

Note that the answers obtained in the two ways are identical.

Test your understanding 2

1 You are interested in buying a book that cost 30 Mnl, but discover that its price has increased by 20%. What is the book's new price?

2 Riverland's GDP of 259 Rvl billion in 2009 grew to 272 Rvl billion in 2010 but then fell to 267 Rvl billion in 2011. Calculate the rate of growth in Riverland's GDP in the period **(a)** 2009–10, and **(b)** 2010–11.

3 In 2005, Riverland had a population of 32.9 million people. In the period 2005–10, its population grew by 7.2%. What was its population in 2010?

4 It is estimated that Flatland's unemployed workers were 1.2 million in 2009. By 2010, the number of unemployed had fallen to 1 million. What was the percentage change in unemployed workers over this period?

5 A company had profits of $2.5 million in 2009. It insists on an 8.0% increase in profits per year. What will its profits be in 2010 if it meets its goal?

2 Understanding and interpreting graphs and diagrams

Graphs and diagrams are simply 'pictures' showing how variables change and how they are related to each other. Sometimes, the information contained in graphs and diagrams can be described in words; however, 'pictures' allow us to see and understand information far more quickly and effectively. Therefore, graphs and diagrams are very important in presenting real-world information as well as in building and presenting economic theories and models.

Graphs that display information about a single variable

A variable is a measure of something that can take on different values; it is something that 'varies' or 'changes'. Very often, we are interested in studying changes in a single variable. We can do this by using three types of graph: pie charts, bar graphs and line graphs. Each one serves different purposes (though there are also some overlaps).

Pie charts

Pie charts are convenient to use when we want to show how a whole is divided up between different parts. The 'raw data' appearing in Table 1 present how the total population in the world in 2008 was divided between the countries in the world according to income levels. The variable is population, which varies according to group of countries. The last column shows the percentage of the world's population that was in each income group. 'High income' countries are those considered to be economically more developed, while 'upper middle income', 'lower middle income' and 'low income' countries are considered to be economically less developed.

This information is presented as a pie chart in Figure 1. The pie chart is derived by multiplying the 360° of a circle by the same percentage that appears for each income level in the table, thus obtaining the 'slice' of the 'pie' that corresponds to each income level. We can see straightaway that a far larger percentage of the world's population, or 84% (=14.2% + 55.3% + 14.5%) lives in economically less developed countries, compared to only 16% who live in more developed ones. Note that Table 1 and Figure 1 display exactly the same information, yet it is much easier to 'read' it in the pie chart.

Table 1 World population (in millions) and distribution among countries grouped by income levels, 2008

Income level	Population (millions)	% of total
High income	1068.5	16.0
Upper middle income	948.5	14.2
Lower middle income	3702.2	55.3
Low income	972.8	14.5
Total	6692.0	100.0

Source: The World Bank, Data Catalog (http://data.worldbank.org/data-catalog).

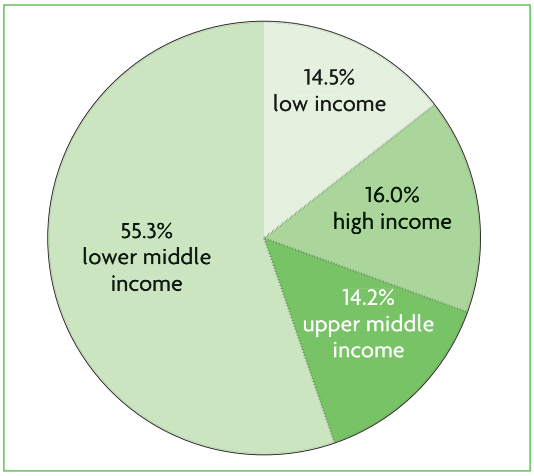

Figure 1 Pie chart: world population (millions) and distribution among countries grouped by income levels, 2008

Source: The World Bank, Data Catalog (http://data.worldbank.org/data-catalog).

Bar graphs

The information in Table 1, presented in a pie chart in Figure 1, can also be presented as a bar graph, as shown in Figure 2. In Figure 2(a), percentages of the world's population are measured along the vertical axis, and the income groups appear along the horizontal axis. Figure 2(b) is the same as Figure 2(a) except that the axes have been reversed. Both presentations are used equally effectively.

However, comparing the pie chart in Figure 1 with the two bar graphs in Figure 2, we can see that the pie chart provides a more effective representation of the different shares of the world's population, because it allows the viewer to see more clearly the relative size of each share of the whole.

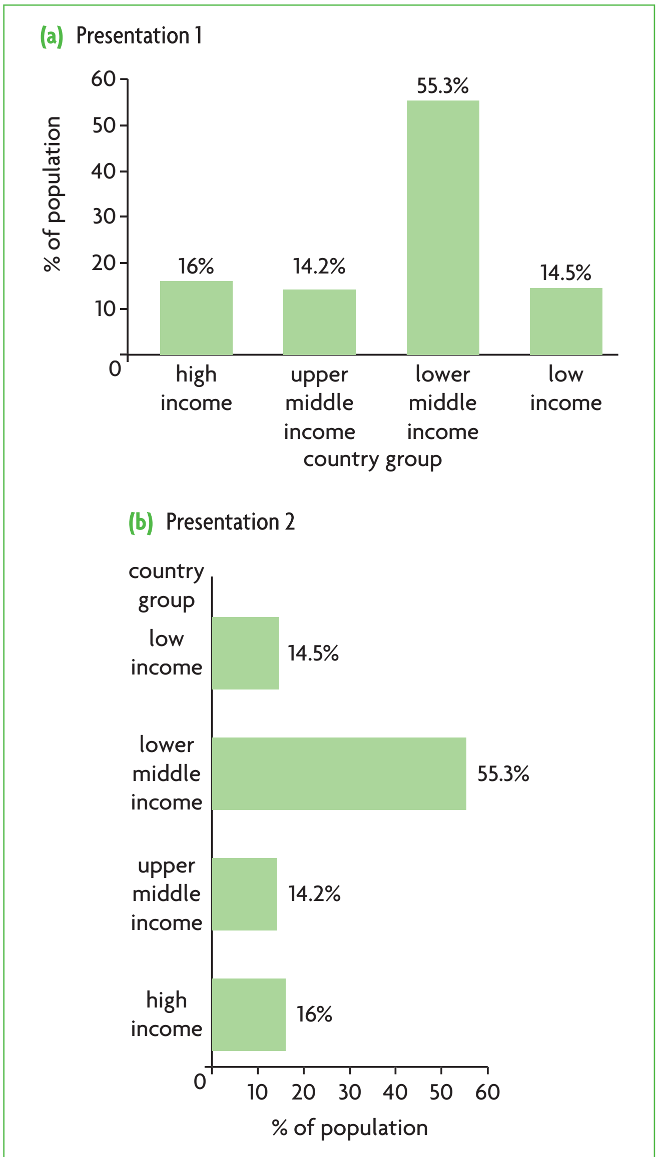

Figure 2 Bar graphs based on the information in Table 1

Bar graphs, on the other hand, are very useful in providing a visual representation of other kinds of information and data that cannot be shown in a pie chart. They are very convenient for illustrating **cross-section data**, which are the values taken by a single variable at a particular time (such as a year) for different groups in a population. The bar graph in Figure 3(a) measures on its vertical axis the percentage of children of primary school age who are enrolled in school, against the countries of the four income groups, which appear on the horizontal axis, in 2007. This graph tells us that 95% of children in high-income countries attend primary school, compared to 77.6% in low-income ones, and so on for the other two groups and the world as a whole.

Bar graphs need not measure percentages on the vertical axis. The bar graph in Figure 3(b) shows the number of tourist arrivals per 1000 people in a given year for selected countries. For example, in Cyprus, there are 2676 tourists per year for every 1000 local residents, in Saint Kitts and Nevis there are 2259, and so on for the other countries shown.

Bar graphs are also useful in presenting how a variable changes for each group from one time period to another. This can be seen in Figure 3(c), which shows the rate of growth of GDP for three years and for seven country groups. This kind of graph allows us to see not only how the variable, in this case the rate of growth in GDP, varies from region to region (highest in countries of East Asia and Pacific and lowest in high income countries), but also, we can see that for all country groups, the rate of growth of GDP was lower in 2008 compared to 2007, and was in some cases lower than in 2006.

Some variables illustrated in bar graphs may sometimes take on negative values, that is, the bars can fall below the horizontal axis. Whether or not this can happen depends of course on the nature of the variable (it could not happen in Figure 3(b) because there cannot be a negative number of tourists). Figure 3(d) shows the current account balance as a percentage of GDP in selected countries (we will study the current account balance in Chapter 14 of the textbook). Very briefly and simply (for the time being), the current account balance is a measure of inflows and outflows of money in a country for particular purposes. If inflows are greater than outflows, there is a positive balance, which appears as a bar above the horizontal axis. If inflows are smaller than outflows, there is a negative balance and this appears as a bar below the horizontal axis.

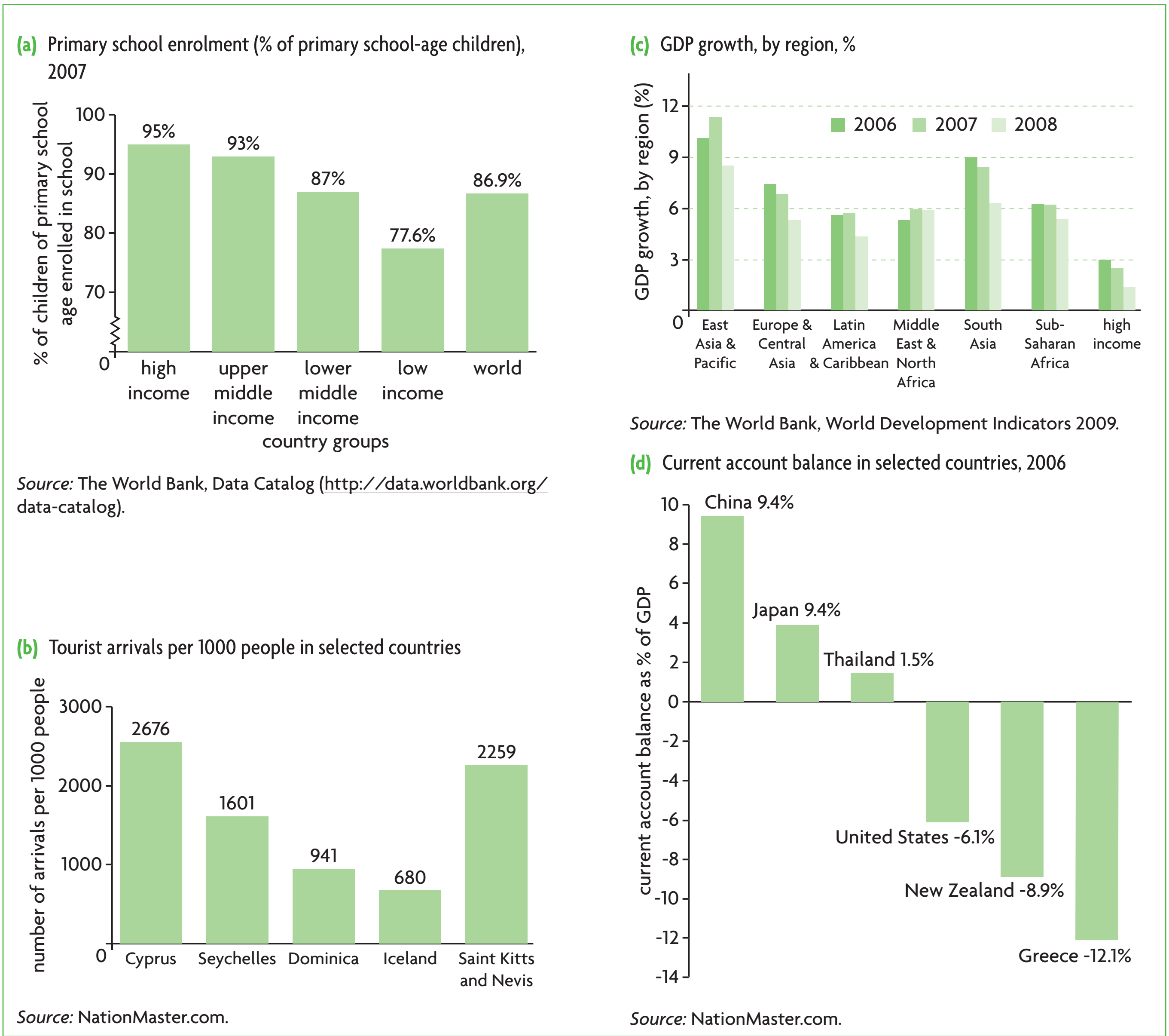

Figure 3 Bar graphs (cross-section data)

Line graphs

Line graphs are a convenient way to represent a variable that takes on different values over time. We have seen that this can to some extent also be shown in a bar graph, as in Figure 3(c). However, if we are interested in seeing how a variable changes over an extended period, and also want to easily make comparisons, a line graph is more appropriate. A line graph typically measures time (in months, years, etc.) on the horizontal axis, while the variable that is being examined appears on the vertical axis. Line graphs showing change over time use **time-series data**, which are data for a variable that changes over time.

Figure 4(a) shows how the unemployment rate (the percentage of unemployed people out of the total labour force) varies over time in the European Union (EU) as well as in two EU countries. The graph covers a period of 28 years, and is very effective in illustrating the fluctuations in the rate of unemployment. In addition, it allows us to make comparisons in unemployment rates of individual countries and the EU, whose unemployment rate is an average over all its members. As we can see, Cyprus's unemployment rate has been lower and Belgium's higher (until about 2003–4) than the average over the EU.

The variables illustrated in line graphs can also sometimes take on negative values, as in the case of bar graphs. Figure 4(b) shows the annual rate of growth of agricultural output (per person) in Sub-Saharan Africa for the period 1968–2004. In some years this has been positive (above the horizontal axis) while in others it has been negative (below the

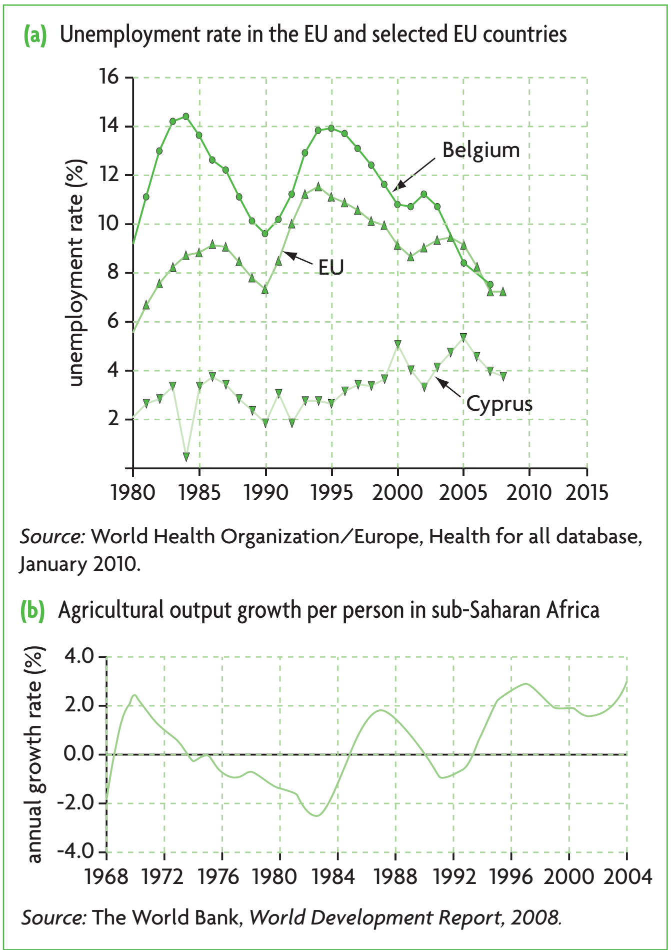

Figure 4 Line graphs (time-series data)

horizontal axis). As you may remember from the discussion above, a positive rate of growth means that output is *increasing*, while a negative rate of growth indicates *decreasing* output.

Test your understanding 3

Look through local newspapers and magazines and find examples of pie charts, bar graphs and line graphs. Which of these display cross-section data; which display time-series data? Explain, in a general way, what each graph illustrates, and describe any patterns or trends you detect.

Graphs displaying the relationship between two variables

Pie charts, bar graphs and line graphs that display information about a single variable are usually constructed in order to present real world data in an organised and logical way, allowing viewers to easily make sense of the information and if possible, detect patterns that increase our understanding of the complicated world around us. Yet very often we want to go beyond a description of the world offered by these graphs in order to discover *how variables are related to each other*. To do this, we must use graphs that focus on two interrelated variables. These kinds of graphs are very important in illustrating economic theories and building economic models.

Constructing and interpreting graphs that relate two variables to each other

Graphs that illustrate how variables are related to each other measure a different variable on each axis. Each axis on the graph represents a number line, which measures units of the variable. Figure 5(a) presents a vertical and a horizontal number line. The vertical number line begins with negative numbers at the bottom end, which increase as we move upward until they reach 0, and then become positive. The horizontal number line begins with negative numbers at the left, which increase as we move rightward until they reach zero, and then become positive. In both number lines, the number 0 is called the *origin*. In each number line, the numbers represent units of the variable that is being measured. Therefore, the value of the variable measured on the vertical number line increases in the upward direction, and the value of the variable measured on the horizontal number line increases in the rightward direction.

To construct a graph, we put the vertical and horizontal number lines together, so that they are perpendicular to each other and intersect at the origin of each one, as shown in Figure 5(b). Each number line in the graph is referred to as an *axis*. By convention, the horizontal axis is sometimes called the *x-axis* and the vertical is called the *y-axis*. The four spaces that are carved out by the intersecting axes are called *quadrants*, and are numbered from I to IV. Quadrant I represents combinations of two positive numbers, quadrants II and IV represent combinations of one positive and one negative number, and quadrant III represents combinations of two negative numbers.

In economics, most of the relationships we examine involve combinations of two positive numbers, in which case we simplify the graph by considering only the first quadrant, ignoring the remaining three. This is shown in Figure 5(c). However, sometimes we may want to examine relationships involving one positive and one negative number, in which case we may consider a graph such as in Figure 5(d) or (e).

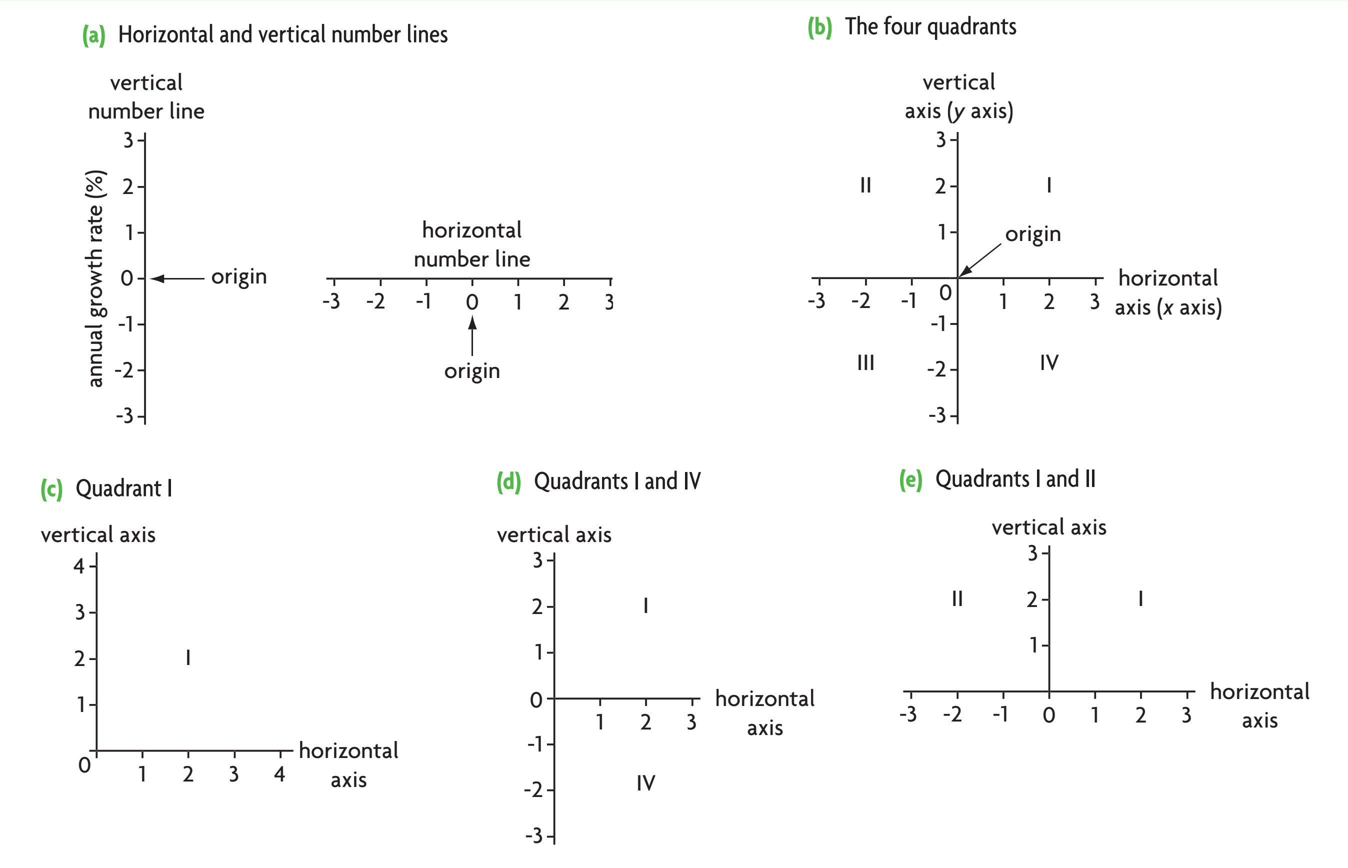

Figure 5 Number lines and quadrants

Positive (direct) relationships between two variables

Each point on a graph is specified by two numbers, one from each of the two axes. Suppose we are examining the relationship between the following two variables: the number of calories consumed per day and the amount of weight (in kilograms, abbreviated as kg) that will be gained in one month. The data on these two variables are presented in Table 2 on page 9, and are graphed (or plotted) in Figure 6(a) where daily caloric consumption is measured on the horizontal axis and monthly weight gain on the vertical axis. Each pair of points in the table corresponds to a point in the graph. For example, point e corresponds to 2500 calories per day, and to 2 kg of weight gain per month. On the graph, this point is found by drawing a line upward from 2500 on the horizontal axis, and also drawing a horizontal line from 2 on the vertical axis; point e is where the two lines meet. Every other point on the graph is plotted in the same way.

Each point in the graph is represented as: (h,v), where h takes on the value of the variable measured on the horizontal axis and v takes on the value of the variable measured on the vertical axis. Therefore, point e is represented as (2500,2); point g as (3000,4).

Each point in a graph can be represented as (h,v), where h = the value of the variable on the horizontal axis and v = the variable on the vertical axis.

Drawing a line through all the points gives a straight line, which is called a **curve**. All lines in graphs (and diagrams) are referred to as curves, regardless whether they are straight or curved.

Note that the horizontal axis contains a squiggly part close to the vertical axis. The reason is that the numbers on this axis jump from 0 to 1500 in a very short space, whereas all other points have an identical distance between them, each space representing 250 calories. The squiggly line means that some numbers of the number line of this axis have been skipped.

Once a line is drawn connecting the points in a graph, it is possible to read off other combinations of the two variables. For example, a point exactly in between points e and f on the curve corresponds to the points between the values of the two variables on each of the axes, which are 2.5 kg and 2625 calories.

The graph shows that consumption of 2000 calories per day results in a steady weight, as weight gain is 0 kg. If calories consumed rise above 2000 per day, the result is weight gain, while if they are less, the result is weight loss (negative 'weight gain' is equivalent to weight loss).

Table 2 The relationship between calories consumed daily and weight gain per month

Daily caloric consumption	Monthly weight gain (kg)	Point on graph (Figure 6(a))
1500	–2	a
1750	–1	b
2000	0	c
2250	1	d
2500	2	e
2750	3	f
3000	4	g

Table 3 The relationship between hours of sleep and mistakes on tests

Number of hours of sleep	Number of mistakes on tests	Point on graph (Figure 6(b))
4	10	a
5	8	b
6	6	c
7	4	d
8	2	e
9	0	f

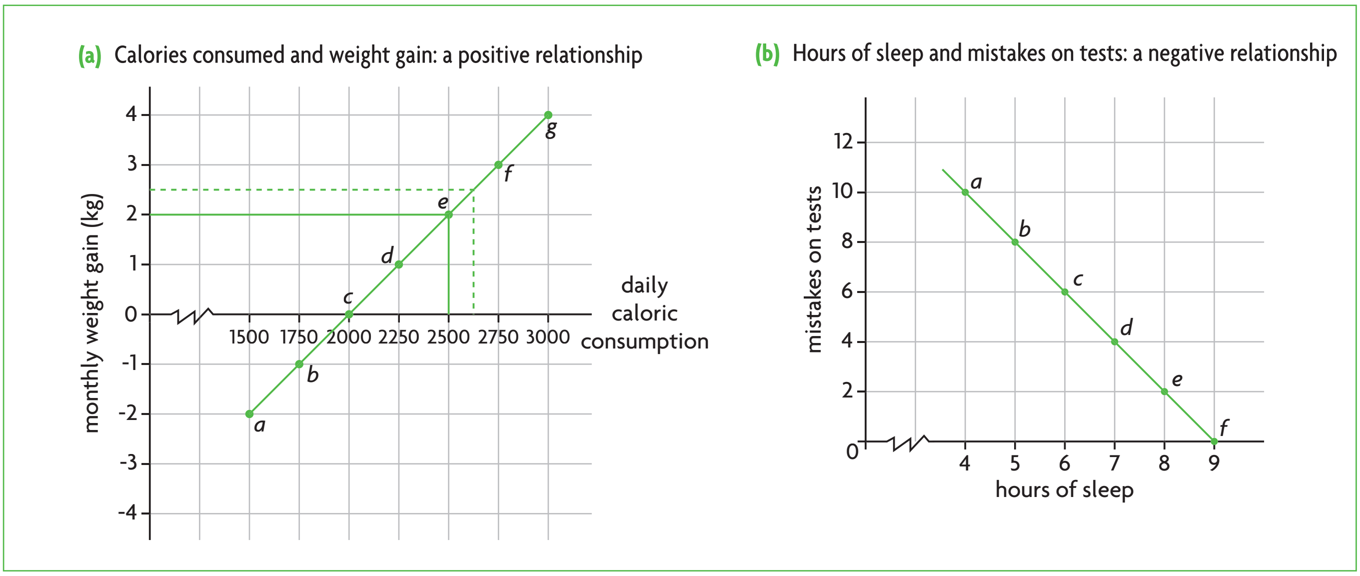

Figure 6 Positive and negative relationships between two variables

The curve of Figure 6(a) shows that the two variables we are examining are related in a particular way: as daily consumption of calories increases, weight gain also increases. This is called a **positive**, or **direct** relationship:

> A positive (or direct) relationship between two variables is illustrated by a curve that moves upward and to the right, showing that as one variable increases, the other also increases. Alternatively, if one variable decreases, the other also decreases. In a positive relationship, the two variables change in the same direction.

Negative (or indirect) relationships between two variables

Let's now consider a different kind of relationship, using the information of Table 3, which provides data on two variables: number of hours of sleep and number of mistakes on tests. Each pair of data corresponds to a single point in the graph that appears in Figure 6(b). This graph is plotted entirely in quadrant I, with positive values of both variables, as it is not possible to have a negative number for hours of sleep or for mistakes on tests. This graph shows that few hours of sleep are associated with a larger number of mistakes on tests, while more hours of sleep mean fewer mistakes on tests. This is called a **negative**, also known as an **indirect**, relationship:

> A negative (or indirect) relationship between two variables is illustrated by a curve that moves downward and to the right, showing that as one variable increases, the other variable decreases. In a negative relationship, the two variables change in opposite directions.

Graphs and the cause-and-effect relationship between two variables

One important reason why we construct and study two-variable graphs is that we are trying to discover **causal relationships** between variables. A causal relationship is a 'cause-and-effect' relationship, where changes in one variable are seen as causing changes in the other variable. The variable that initiates the change is called the **independent variable**, and the variable that is influenced or affected as a result is called the **dependent variable**. A causal relationship is called a **functional relation**, where the dependent variable is a function of the independent variable. Two-variable graphs usually (though not always) display such functional relations.

In Figure 6(a), the independent variable (or the 'cause') is the daily consumption of calories, and the dependent variable (or the 'effect') is monthly weight gain. Monthly weight gain *depends* on consumption of calories, hence is the dependent variable. Figure 6(a) shows a **positive causal relationship**. In Figure 6(b), the independent variable is the number of hours of sleep; in this relationship, mistakes on tests *depend* on hours of sleep. In Figure 6(b), we see a **negative causal relationship**.

Two-variable graphs usually represent a causal relationship between the variables, where the independent variable is the 'cause' and the dependent variable is the 'effect'. Causal relationships are very important for making hypotheses, constructing theories that try to explain events and as building blocks of models that illustrate theories.

Test your understanding 4

For each of the following pairs of variables, explain (i) whether there is likely to be a positive or negative causal relationship between them, and (ii) which is the dependent and which is the independent variable:

(a) income and saving

(b) number of DVDs purchased and price of DVDs

(c) level of salary and number of years of working experience

(d) the temperature and number of swimmers on the beach.

Causation versus association (correlation)

Two variables may sometimes appear to be related to each other, and yet there may not be a causal relationship between them. There are three important cases leading to difficulties.

Case 1: We may observe that a society has a large number of doctors (*per capita* or per person in the population) and it also has high rates of certain diseases. We could conclude that the high rate of diseases has given rise to a large number of doctors (though we could also conclude the opposite). But would this be a valid argument? It is very possible (and likely) that each of these (the large number of doctors and the high rate of diseases) has separate, independent causes, and the apparent relation between the two is coincidental.

Case 2: A second possible difficulty is illustrated by the following simple example. Runny noses are observed to appear together with sore throats. We therefore conclude that a runny nose causes a sore throat (or vice versa). But this is clearly nonsense. Both runny noses and sore throats are caused by another (third) factor, which is usually a virus. In this case, the association of runny noses and sore throats is usually not coincidental, but the two are not causally related to each other.

Case 3: A third difficulty arises in the event that there may be a causal relationship between two variables, but we cannot be sure which causes which. For example, it is generally observed that people with more education also have higher incomes. It may therefore be supposed that more education allows people to get better jobs and therefore earn higher incomes. However, the opposite is also possible, as people with higher incomes have greater possibilities to become educated. So which causes which?

In the first two cases, there is an **association** (in mathematical terms this is called a **correlation**) between the two variables, but with no causation. In the first case, this is due to coincidence; in the second, it is due to there being another causal factor which affects both variables. In the third case, the association (or correlation) may well be due to causation, but we cannot be sure which is the causal factor.

The cases of demand and supply

In your study of economics, you will encounter many kinds of relationships between variables that are illustrated as graphs or diagrams. Two very important relationships you will make great use of are those of demand and supply.

Demand

Demand as a relationship between price and quantity demanded

Demand involves the relationship between the price of a good and the quantity of the good consumers want to buy. Table 4 shows the quantity of pizzas that the residents of Olemoo want to buy at different possible prices (in Olemooan $) each week. The data in Table 4 are graphed (or plotted) in Figure 7, where we see that there is a negative (or indirect) relationship between price and quantity: the higher the price, the fewer the pizzas that Olemooans want to buy. The quantity of pizzas Olemooans want to buy at each price is called **quantity demanded**.

Note that of the two variables we are considering, price and quantity demanded, price is the independent variable and quantity demanded is the dependent variable. This is because the quantity of pizzas Oloemooans want to buy *depends* on the price of pizzas. Therefore, there is a **negative causal relationship** between price and quantity demanded. According to correct mathematical practice, the independent variable is plotted on the horizontal axis and the dependent variable on the vertical axis (as in the cases of Figures 6 (a) and (b) above). However, many economics graphs do not follow the customary mathematical practice. *In the case of demand, the independent variable 'price' always appears on the vertical axis, while the dependent variable, 'quantity', always appears on the horizontal axis.*

Table 4 Demand for pizzas by Olemooans

Price per kg (Olemooan $)	Quantity demanded (thousand kg per week)
1	9
2	7
3	5
4	3
5	1

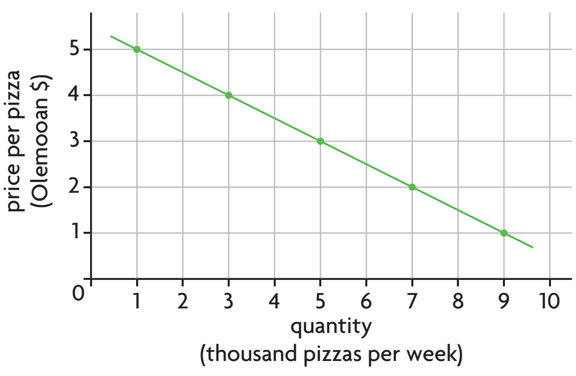

Figure 7 Demand curve: price of pizzas and quantity demanded

Shifts of the demand curve and the ceteris paribus assumption

So far, we have studied the relationship between two variables only. However, in the real world, a dependent variable usually depends on more than just one independent variable. For example, the amount of pizzas that Olemooans want to buy very likely depends not only on the price of pizzas but also on Olemooans' income, Olemooans' tastes (how much they like pizza), the number of people in the Olemooan population, and other factors. Taking income as an example, it is likely that as Olemooans' income increases, they will want to buy more pizzas. Yet this complicates matters. What if income increases, and at the same time the price of pizzas also increases? Will Olemooans want to buy more or fewer pizzas?

We now have three variables, but with the possibility of showing only two of these at the same time in a graph of a demand curve. To resolve this problem, we plot the relationship between price and quantity demanded *on the assumption that income (plus all other things that can affect demand for pizzas) is constant or unchanging*. This is called the *ceteris paribus* assumption (explained also in Chapter 1 of the textbook, page 10), which means that all other things that can affect a relationship between two variables are assumed to be constant or unchanging.

To deal with the likely effects of income on pizzas demanded in a graph, we use the information in Table 5, which contains data on quantities of pizzas demanded for different levels of income. The data in the column indicating 'middle income', are the same as those used to plot the demand curve in Figure 7.

Figure 8 graphs the demand curves corresponding to each of the three income levels. We can see that if Olemooans at first have a middle income (the middle curve), but then their income increases, the entire demand curve shifts (or moves) to the right, to the curve labelled 'high income'. This curve tells us that for any particular price, Olemooans will buy more pizzas with a higher income, and is called 'an increase in demand'. If, however, Olemooans' income falls, their demand curve will shift to the left to the curve labelled 'low income'; this is called a 'decrease in demand'. This means that for a particular price, Olemooans will buy fewer pizzas.

Leftward/rightward shifts and upward/downward shifts of the demand curve

We have referred to the demand curve as shifting 'to the right' or 'to the left', which is the customary practice. However, if you examine Figure 8, you will notice that a rightward shift looks the same as an upward shift, and a leftward shift looks exactly the

Table 5 Demand for pizzas by Olemooans at different income levels

Price per kg (Olemooan $)	Quantity bought (thousand pizzas per week) Low income	Quantity bought (thousand pizzas per week) Middle income	Quantity bought (thousand pizzas per week) High income
1	8	9	10
2	6	7	8
3	4	5	6
4	2	3	4
5	0	1	2

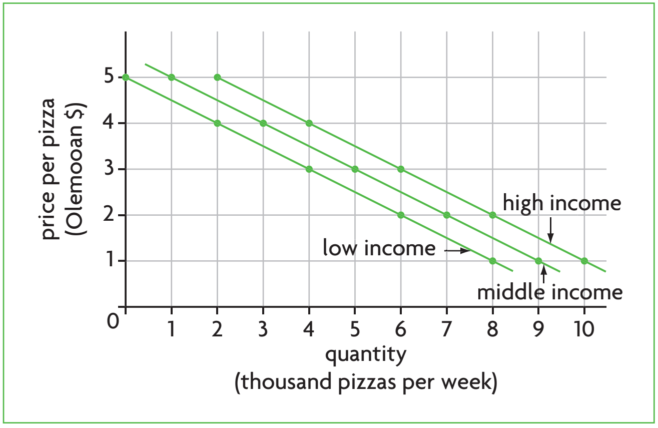

Figure 8 Demand curves at different income levels

same as a downward shift. *The meaning of a rightward shift of a demand curve is exactly the same as an upward shift, and the meaning of a leftward shift of a demand curve is the same as a downward shift*, although there is a difference in how we can interpret them.

If we view the curve as shifting to the right, we see that for a given price, Oloemooans increase their purchases of pizzas as income rises. At a price of $3, they will buy 4000 pizzas per week with a low income, 5000 pizzas with a middle income, and 6000 pizzas with a high income. If we view the curve as moving upward or downward, we see how much Olemooans are willing to pay for a particular quantity of pizzas as their income changes. For example, for the quantity of 6000 pizzas, they are willing to pay $2 per pizza with low incomes, $2.50 per pizza with medium incomes, and $3.00 per pizza with high incomes.

Usually, when studying demand curves, we examine shifts as occurring in the rightward or leftward directions, and less often in the upward and downward directions.

Distinguishing between a movement along the demand curve and a shift of the curve

It is very important to make a distinction between **a movement along a demand curve**, and a **shift of a demand curve**. In a causal relationship, a movement along a curve can only be caused by a change in the independent variable (in this case price), which influences the dependent variable (quantity demanded), thus causing a movement from one point on the curve to another. A shift of a curve, on the other hand, is caused by a change in a variable that was previously held constant under the *ceteris paribus* assumption (the level of income). All variables that can cause shifts of a demand curve are referred to as **determinants of demand**, because they *determine* the position of the demand curve. As you will learn in Chapter 2 of the textbook, there are several important determinants of demand, of which income is only one.

Supply

Supply as a relationship between price and quantity supplied

Supply involves the relationship between the price of a good and the quantity of the good producers (or firms) want to produce and sell. Table 6 shows the quantity of pizzas Olemooan pizza producers want to produce each week at different possible prices (in Olemooan $). The data in Table 6 are graphed in Figure 9 (see page 13), which shows that there is a positive (or direct) relationship between price and quantity: the higher the price of pizzas, the more pizzas that Olemooan pizza producers want to sell. The quantity of pizzas Olemooan producers want to sell is called **quantity supplied**.

As in the case of demand, price is the independent variable and quantity supplied is the dependent variable, because the quantity supplied *depends* on price. There is therefore a **positive causal relationship** between price and quantity supplied. Again, as in the case of demand, *supply curves measure the independent variable, price, on the vertical axis and the dependent variable, quantity, on the horizontal axis*.

Shifts of the supply curve and the ceteris paribus assumption

The amount of a good produced, such as pizzas, depends on more factors than just price. For example, it depends on the number of pizza producers; as this number increases, the amount of pizzas produced will also increase. In addition, it depends on the costs of producing pizzas; if costs increase, it is likely that the amount of pizzas produced will fall.[1]

[1] The reasons for this are explained in Chapter 2 of the textbook.

Table 6 Supply of pizzas by Olemooan producers

Price per pizza (Olemooan $)	Quantity supplied (thousand pizzas per week)
1	3
2	4
3	5
4	6
5	7

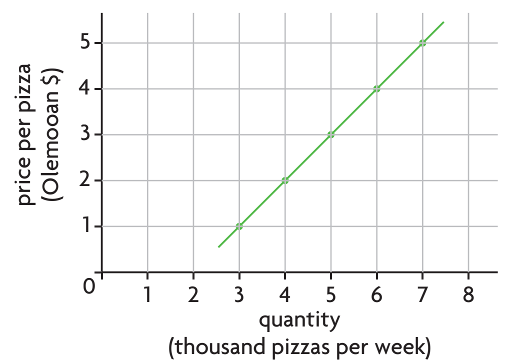

Figure 9 Supply curve: price of pizzas and quantity supplied

Once again, as in the case of demand, we have more than two variables to deal with, but with the possibility of showing only two of them at the same time in a single supply curve. To address this problem we use the same method as with demand, which involves use of the *ceteris paribus* assumption. We plot the relationship between price and quantity supplied, on the assumption that all other variables that can influence the amount supplied are constant or unchanging.

We can use the information in Table 7 to show what happens to the supply curve when there are factors other than price that affect the amount supplied. The factor we will consider is the cost of producing pizzas. The data in the column indicating 'medium costs' are the same as those we used to plot the supply curve of Figure 9.

Figure 10 graphs the supply curves corresponding to each of the three cost levels. If Olemooan producers have medium costs of production for pizzas and then these costs decrease, the entire supply curve shifts to the right, to the curve indicating 'low costs'. This is called an 'increase in supply', and tells us that at each possible price, producers will supply more pizzas than before. If, however, costs of producing pizzas increase, then the supply curve will shift to the left to the curve labelled 'high costs', meaning that at each possible price producers want to supply fewer pizzas than before.

Leftward/rightward shifts and upward/downward shifts of the supply curve

Looking at Figure 10, we can see that a rightward shift of the supply curve looks the same as a downward shift, and a leftward shift looks the same as an upward shift. *The meaning of a rightward shift of a supply curve is exactly the same as a downward shift, and the meaning of a leftward shift of a supply curve is exactly the same as an upward shift*, though it is possible to interpret them differently (as in the case of demand).

Viewing the curve move from left to right, we see that for each price, pizza producers want to supply more pizzas as their costs fall. At a price of $3, they will supply 4000 pizzas when costs are high, 5000 when costs are medium, and 6000 when costs are low. Viewing the curve move upward, we see what price producers are willing to accept for pizzas as their costs change. To produce 5000 pizzas, for example, they will want to accept a price of $2 if costs are low, $3 if costs are medium, and $4 if costs are high.

Table 7 Supply of pizzas by Olemooan producers at different cost levels

Price per pizza (Olemooan $)	Quantity supplied (thousand pizzas per week) Low costs	Quantity supplied (thousand pizzas per week) Medium costs	Quantity supplied (thousand pizzas per week) High costs
1	4	3	2
2	5	4	3
3	6	5	4
4	7	6	5
5	8	7	6

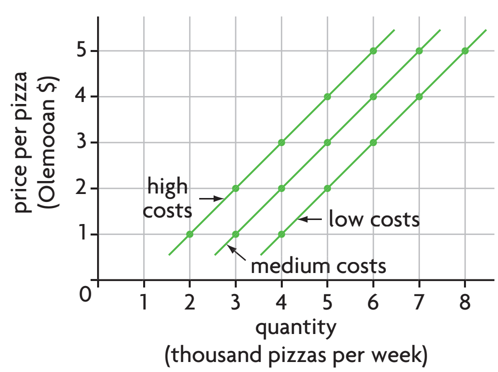

Figure 10 Supply curves at different cost levels

This is reasonable, because the higher the costs of making pizzas, the higher must be the price to make it worthwhile for producers to produce the pizzas.

Supply curve shifts are sometimes examined as moves to the left or to the right (see Chapter 2 of the textbook) and sometimes as moves up or down (textbook Chapters 4 and 5). However, it is important to note that *when we speak of 'an increase in supply' or 'a decrease in supply' we are always referring to leftward/ rightward shifts*. An increase in supply means that more pizzas are sold at each price, while a decrease in supply refers to fewer pizzas sold at each price. Since we measure the amount of pizzas along the horizontal axis, this means that supply increases or decreases involve rightward or leftward shifts of the supply curve.

Why, then, do we sometimes refer to upward and downward shifts of the supply curve? As you will discover when you study Chapters 4 and 5, upward/ downward shifts are very convenient to use when studying certain kinds of taxes and subsidies, and their effects on firms' supply curves. Taxes work to increase firms costs' of production, and as we know this results in a leftward shift of the supply curve (or a 'decrease in supply'). But *if we view this leftward shift as an upward shift, then it is actually possible to measure the cost increase in our graph*, which can be very useful. Subsidies, on the other hand, are payments by the government to firms, having the opposite effects of taxes. They work to decrease firms' costs of production, resulting in a rightward shift of the supply curve (or an 'increase in supply'). *If we view this rightward shift as a downward shift, we can again measure the decrease in production costs*, which can also be very useful.

These points will become clearer to you as you study the textbook.

Distinguishing between a movement along the supply curve and a shift of the curve

Everything that was said above in connection with shifts versus movements along a demand curve applies equally to the supply curve. A movement along a supply curve is caused only by changes in price, the independent variable. A shift of a supply curve is caused by changes in variables previously held constant under the *ceteris paribus* assumption. All such variables are called **determinants of supply**, because they determine the position of the supply curve. In Chapter 2, you will learn about several different kinds of determinants of supply.

This leads us to an important point that applies to both demand and supply curves (as well as all other curves):

A shift of a curve can be caused only by changes in variables that *do not appear* on the vertical or horizontal axis of a graph. Such variables are called *determinants* of the curve, because they determine the position of the curve on the graph. Determinants include all the variables that are held constant under the *ceteris paribus* assumption. If any determinant changes, then the entire curve shifts. On the other hand, any change caused by a variable that is plotted on the vertical or horizontal axis, always leads to a movement along the curve.

Test your understanding 5

1 Draw a demand curve for pizzas (it is not necessary to use data) and show what is likely to happen if **(a)** there is a change in the price of pizzas, **(b)** there is an increase in the population of Olemoo, and **(c)** there is a decrease in the population of Olemoo.

2 Draw a supply curve for pizzas (it is not necessary to use data) and show what is likely to happen if **(a)** there is a change in the price of pizzas, **(b)** there is an increase in the number of pizza producers, and **(c)** there is a decrease in the number of pizza producers.

Further topics on graphs

Non-linear curves

The curves discussed above, illustrating positive and negative relationships, are *linear*, or straight-line curves. Many of the curves you will be studying in this course at standard level will be linear. Higher level material will include some curves that are *non-linear*, or curved. Everything that was said above regarding positive and negative relationships applies to non-linear curves as well.

Figure 11(a) shows two non-linear curves where the variables in both cases are positively (directly) related. The difference between them is that in (i), as we move rightward along the x-axis the curve becomes steeper, where as in (ii), the curve becomes less steep.

In Figure 11(b), the two variables are negatively (indirectly) related to each other. Here, too, the difference between the two curves lies in how their steepness changes as we move rightward along the x-axis: in (i) steepness decreases and in (ii) steepness increases. The curve in Figure 11(b)(ii) is the production possibilities curve introduced in Chapter 1 of the textbook.

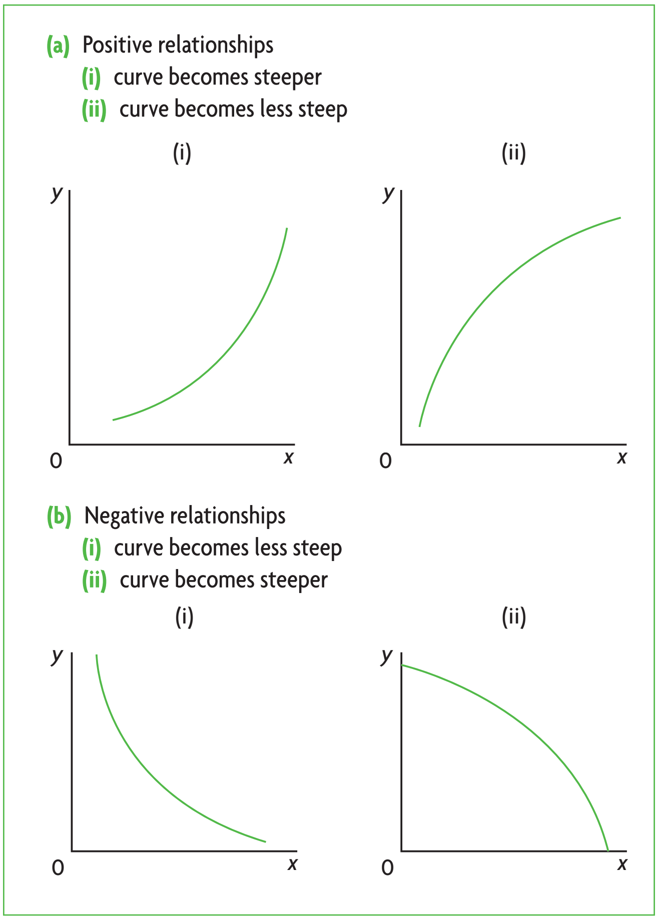

Figure 11 Non-linear curves

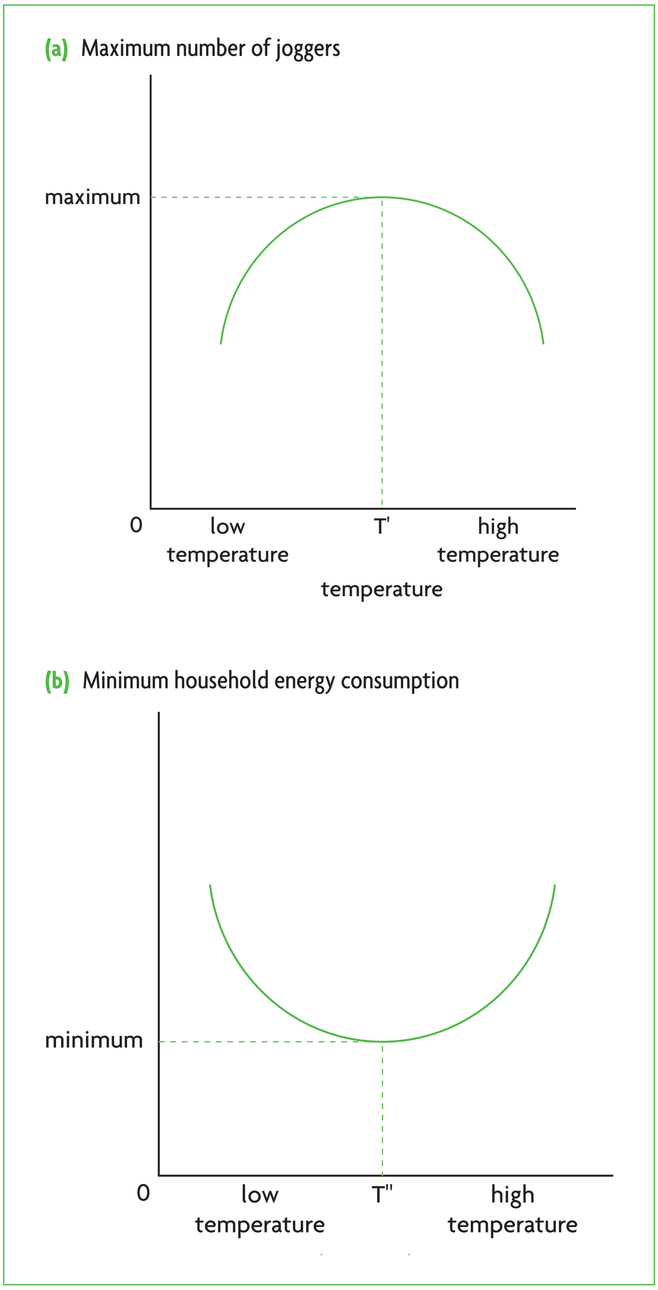

Figure 12 Non-linear curves with maximum and minimum points

Maximum and minimum points of variables in non-linear curves

Sometimes a variable that is plotted in a non-linear relationship reaches a *maximum*, which is a 'highest' point, or a *minimum*, which is a 'lowest' point. These kinds of points are illustrated in Figure 12. In part (a) temperature is measured on the horizontal axis, and the number of joggers on the vertical axis. When it is very cold there are few joggers; as temperature increases, the number of joggers increases, but beyond a certain temperature, *T*′, it gets too hot to jog, and the number of joggers begins to fall. At temperature *T*′, the number of joggers is maximum. Therefore, the curve up to *T*′ shows a positive relationship, and beyond *T*′ it becomes a negative relationship.

In part (b) the horizontal axis again measures temperature, with household energy consumption appearing on the vertical axis. At low temperatures energy consumption is high because of heating, and at high temperatures it is high because of air conditioning. There is a temperature in between the two extremes, *T*″, where energy consumption is minimum because there is no need for heating or air conditioning. The curve up to *T*″ shows a negative relationship between the two variables, and beyond *T*″ it becomes a positive relationship.

Illustrating two variables that are not related to each other

Sometimes, we may run into variables that are not related to each other. This means that as one variable changes, the other variable remains the same. Two such relationships are shown in Figure 13. In part (a), we see that as fruit consumption increases, the

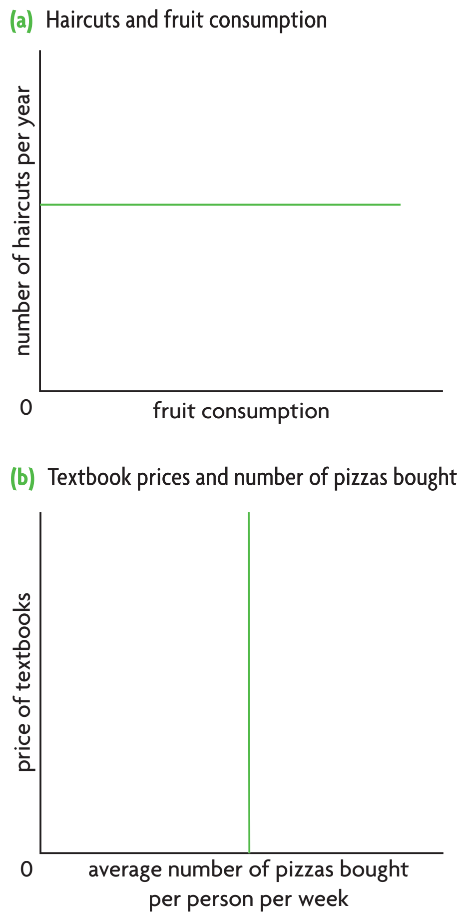

Figure 13 Unrelated variables

number of haircuts per year remains constant, i.e. fruit consumption does not have any effect on the number of haircuts. In part (b) we see that the number of pizzas bought per person each week remains constant as the price of textbooks changes; in other words, textbook price changes have no effect on pizza consumption. In both these examples, we say the two variables are *independent of each other*.

Calculating areas in a graph

In some situations, some areas in a graph may have a particular meaning that we may want to calculate. For example, an important concept in economics is total revenue (abbreviated as *TR*), which is the total amount of income that a firm receives for selling its output. Total revenue is calculated by multiplying the number of items of a good sold, or quantity (*Q*), by the price (*P*) of the good. Therefore, $TR = P \times Q$. We can see this graphically in Figure 14, which plots price on the vertical axis and quantity on the horizontal axis. The curve shown is a supply curve (*S*), indicating that the quantity of a good sold by the firm increases as the

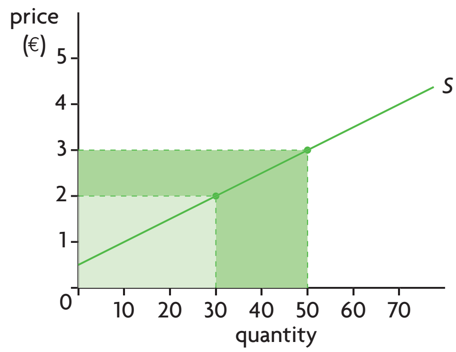

Figure 14 Calculating areas in a graph

price of the good increases (see page 12). We can see that when price is €2 per unit, the firm sells 30 units; when the price rises to €3 per unit, the firm sells 50 units. What is the firm's total revenue (*TR*) at each price? Since $TR = P \times Q$, when price is €2, $TR = €2 \times 30 = €60$, and when price is €3, $TR = €3 \times 50 = €150$. These values for *TR* can be shown graphically as areas. Since the area of a rectangle can be found by multiplying two of its adjacent sides (two sides that connect at a point), it follows that $TR = €60$ is shown by the light green rectangle, and $TR = €150$ is shown by the sum of the dark plus light green shaded regions. Note that the dark green area alone is the difference between the two, or €90 (=€150 – €60).

Test your understanding 6

In Figure14, assume that the price increases to €4 per unit. **(a)** What quantity of output will be sold at that price? **(b)** What is the firm's total revenue at the new price? **(c)** By how much has the firm's total revenue increased compared to when the price was €3 per unit?

Graphs and diagrams in relation to theories and models

The terms 'graph' and 'diagram' are often used interchangeably, and although their meanings certainly overlap, they are not identical. 'Diagram' is a broader term than 'graph', as all graphs are diagrams, yet not all diagrams are graphs. A diagram is any two-dimensional representation of information, which may be non-numerical, i.e. may not involve numbers (although there are many exceptions). A graph, in a most general sense, is a type of diagram that usually displays variables using numbers or quantities (though here, too, there are many exceptions). Graphs

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are extremely useful in presenting and analysing numerical information, or statistics or data.

Many of the figures presented above consist of diagrams that are graphs, as they all present variables that take on numerical values. On the other hand, the diagrams in Figures 11–13 would usually not be called 'graphs', because they present information about variables in a non-numerical way.

In economics, when we discuss theories and use models to illustrate theories, we make very heavy use of diagrams that show relationships between variables, but without the use of numbers to measure different values of the variables. In fact, it is not necessary to plot data to show how variables relate to each other. For example, the relationship between price (P) and quantity supplied (Q) could be redrawn as in Figure 15(a); the relationship between price (P) and quantity demanded (Q) could be shown as in Figure 15(b). Neither of these diagrams presents any numerical information, yet we can still immediately see the positive relationship in the first diagram and the negative relationship in the second. Most often, the shape or steepness of curves, either on their own, or drawn together with other curves in the same diagram, are all we need in order to be able to make use of the information they provide to develop theories and illustrate them with models.

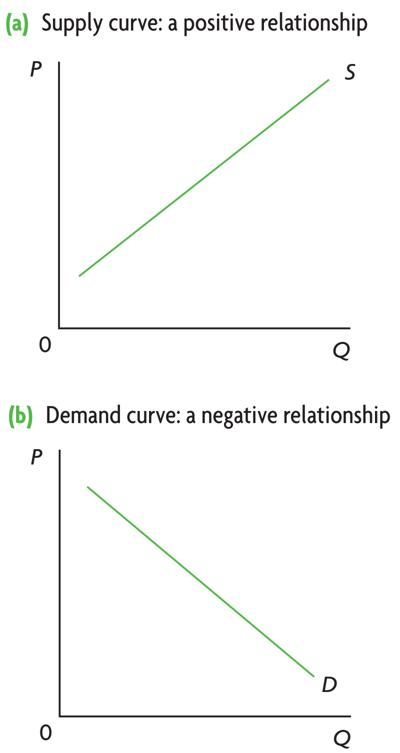

Figure 15 Non-numerical diagrams showing relationships between variables

3 Linear functions and linear equations

In this section we examine graphs of supply and demand and the equations that describe them, as well as some of their properties.

In a functional relation between two variables, the dependent variable is a **function** of the independent variable: the dependent variable changes in response to changes in the independent variable. A function is expressed as an equation. In this course, we will be using only linear functions, or equations that represent straight (not curved) lines in a graph.

According to mathematical convention, the dependent variable in a function is plotted on the vertical axis and the independent variable on the horizontal axis. This practice is followed by the sciences and social sciences including economics, *with one major exception: demand and supply functions, where the axes for dependent and independent variables are reversed*. The practice of reversing the axes dates back to the famous British economist Alfred Marshall, who adopted the custom of plotting price (the independent variable) on the vertical axis in the 19th century. *These functions must therefore be interpreted differently from the standard mathematical way*.

The supply function

Understanding the equation of a supply function

You may recall that supply involves a positive causal relationship between price and quantity supplied. The data in Table 8 represent a positive, linear relationship between price, P, in \$, and quantity supplied, Q_s, in thousand units of good Alpha supplied per day. The data in Table 8 are graphed in Figure 16(a), where you can clearly see the positive relation between P and Q_s.

Table 8 Data for a supply curve: positive linear relationship

Quantity supplied (Q_s, in thousand units of Alpha per day) (dependent variable)	Price (P, in \$) (independent variable)	Point on graph (Figure 16(a))
5	0	A
10	20	B
15	40	C
20	60	D
25	80	E
30	100	F

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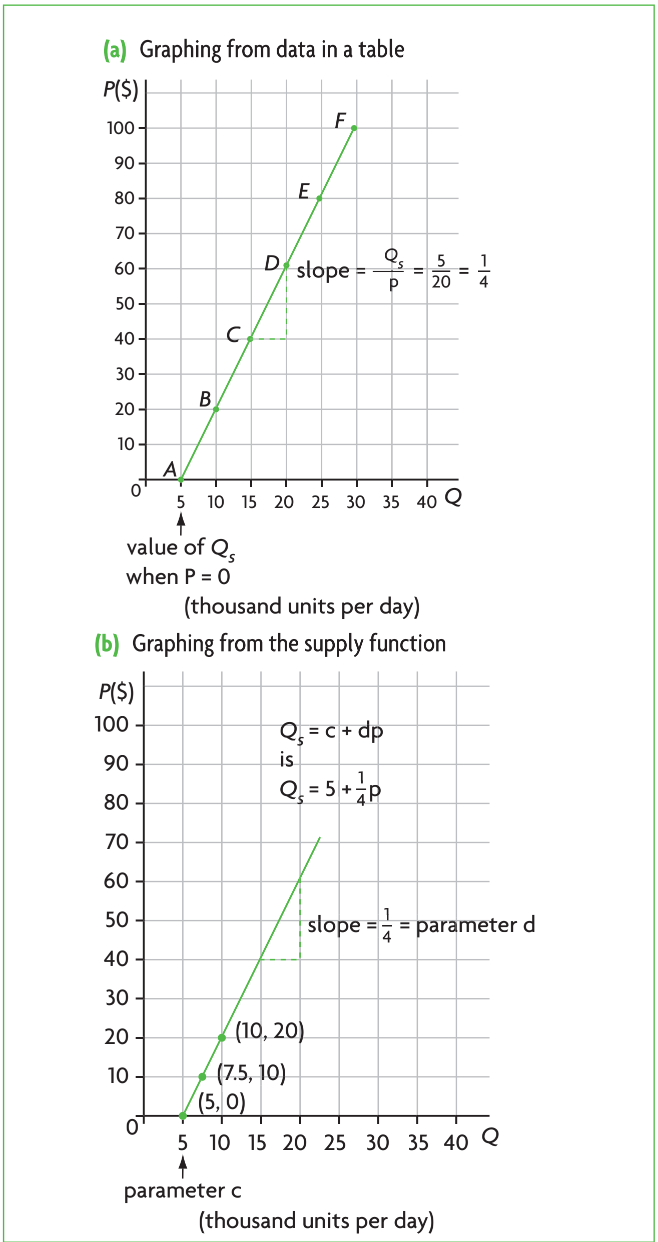

Figure 16 Graphing the supply curve

The equation of a supply function (positive linear function)

The equation of a linear supply function is given by:

$$Q_s = c + dP$$

where

Q_s = quantity supplied, the dependent variable

P = the independent variable

c = the value of the dependent variable when the independent variable, P, is equal to zero

d = the slope, where slope $= \frac{\Delta Q_s}{\Delta P}$ and is the coefficient of the independent variable, P.

Both c and d are known as **parameters**, and in this context refer to the quantities that define a particular linear functional relation. The information in Table 8 determines the values of the parameters c and d, and once we have the values for c and d, we will have the linear equation that describes the data in this table.

The value of c is the value of the dependent variable when the independent variable is equal to zero

In Table 8, the dependent variable is Q_s and the independent variable is P. We can see immediately that when P is equal to zero, Q_s is equal to 5. Therefore, c = 5.

The value of d is the slope

The slope is defined as the change in the dependent variable divided by the change in the independent variable, between any two points on a line. We can calculate the slope from the information in Table 8. Since Q_s is the dependent variable and P is the independent variable, the slope is defined as:

$$\text{slope} = \frac{\Delta Q_s}{\Delta P} = \frac{Q_1 - Q_2}{P_1 - P_2}$$

where Q_1 and Q_2 are two quantities and P_1 and P_2 are the corresponding two prices. Note that this is the same as:

$$\text{slope} = \frac{\Delta Q_s}{\Delta P} = \frac{Q_2 - Q_1}{P_2 - P_1}$$

However, you must be consistent; i.e.

slope $= \frac{Q_2 - Q_1}{P_1 - P_2}$ would be wrong.

Taking points A and B, we have:

$$\Delta Q_s = 10 - 5 = 5, \text{ and } \Delta P = 20 - 0 = 20.$$

Therefore,

$$d = \text{slope} = \frac{\Delta Q_s}{\Delta P} = \frac{5}{20} = \frac{1}{4}$$

If we use points C and D, we have $\Delta Q_s = 20 - 15 = 5$, and $\Delta P = 60 - 40 = 20$. Therefore, $\frac{\Delta Q_s}{\Delta P} = \frac{5}{20} = \frac{1}{4}$, which is the same as before. In fact, *the slope of a straight line is constant*, or the same, no matter which two points we use to calculate it. You can see this by finding the slope between any two other points in Table 8.

The fact that the slope in the supply function $Q_s = c + d\,P$ has a positive sign (+d) indicates that there is a positive relationship between the two variables, P and Q_s.

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Specifying the supply function

Since c = 5, and $d = \frac{1}{4}$, the equation for the supply function in Table 8 is:

$$Q_s = 5 + \frac{1}{4}P$$

Using this equation, we can find any value of Q_s, given a value of P. Suppose $P = 52.7$:

$$Q_s = 5 + \frac{1}{4}(52.7) = 5 + 13.2 = 18.2$$

Similarly, we can find any value of P, given a value of Q_s. Suppose $Q_s = 25$:

$$25 = 5 + \frac{1}{4}P \quad \Rightarrow \quad 20 = \frac{1}{4}P \;\Rightarrow\; P = 80.$$

The equation of a linear supply function is given by $Q_s = c + dP$, where Q_s is quantity supplied (the dependent variable), P is price (the independent variable), c is the value of Q_s when $P = 0$, and d is the slope (given by $\frac{\Delta Q_s}{\Delta P}$).

The positive sign of the slope (+d) indicates that the supply function represents a positive (direct) relationship between P and Q_s.

Graphing the supply curve

We can graph (or plot) the curve that corresponds to this linear function by plotting the data of Table 8. However, *you need to know how to graph the curve by using the equation* $Q_s = 5 + \frac{1}{4}P$ *without the use of data*.

To make a graph, we must first decide which variable will be plotted on which axis. This is very important, because *depending on which variable is plotted on which axis, we get a different linear curve*.

Since we are graphing a supply function, we know (from our discussion above) that *price,* P *(the independent variable), must be measured on the vertical axis, and quantity supplied,* Q_s *(the dependent variable), on the horizontal axis*. We begin by labelling the axes correctly. Price on the vertical axis is measured in $, and quantity on the horizontal axis is measured in 'thousand units per day'.

To find the linear curve that corresponds to our function (equation), we need to find at least two points on the curve, since any two points define a straight line. In practice, it may be a good idea to find a third point to check our calculations (if we find that joining three points does not produce a straight line, we know that we have made a mistake). To find points, we can set P equal to different values and solve for Q_s (though it is possible also to set Q_s equal to different values and solve for P).

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Using our supply equation

$$Q_s = 5 + \frac{1}{4}P$$

it is simplest to begin by setting $P = 0$, which gives $Q_s = 5$, where 5 is the parameter c. This gives us the point (5,0) on the line. To find a second point, we can set

$P = 10$, which gives $Q_s = 5 + \frac{1}{4}(10) = 5 + 2.5 = 7.5$;

this gives us the point (7.5,10). Setting $P = 20$, we have $Q_s = 5 + \frac{1}{4}(20) = 5 + 5 = 10$, or the point (10,20).

We could go on finding more points; however, as explained above, this is not necessary. Now the points can be plotted. They appear in Figure 16(b).

When we have a graph of the general supply function $Q_s = c + d\,P$, we can attach specific interpretations to the parameters c and d. We know from our discussion above that c is equal to the value of the dependent variable, Q_s, when the independent variable, P, is equal to zero. When the dependent variable is plotted on the horizontal axis, as in the supply function, we can re-phrase this to mean that c is the point on the horizontal axis that is cut by the line. This is known as the **horizontal intercept** (an intercept is a point on an axis that is cut by the graphed line). You can see in Figure 16 that the horizontal intercept is the point (5,0).

The parameter d, or slope, is the coefficient of the independent variable, P, defined as $\frac{\Delta Q_s}{\Delta P}$. Since Q_s is plotted on the *horizontal* axis, $\frac{\Delta Q_s}{\Delta P}$ refers to the *horizontal change divided by the vertical change* between any two points on the line.

When the dependent variable appears on the horizontal axis, in the equation of a supply function of the form $Q_s = c + dP$, the parameter c is equal to the horizontal intercept, and the parameter d, or slope, is equal to the horizontal change divided by the vertical change.

An important note on the slope

As noted above, the practice of reversing the axes of the dependent and independent variables gives rise to different graphs. It also gives rise to different

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interpretations of the parameters c and d in the function $Q_s = c + dP$. Suppose that according to the correct mathematical practice, P, the independent variable, is plotted on the horizontal axis, and Q_s, the dependent variable, on the vertical axis. In this case, the parameter c is the **vertical intercept**, or the point on the vertical axis that is cut by the line, and the parameter d, or slope, is equal to the vertical change divided by the horizontal change between any two points on the line, commonly known as 'rise over run'. (This is most likely the interpretation of the slope that you are familiar with.) You can try plotting the graph to see that the linear curve will be very different from that graphed in Figure 16. However, in the context of economics, *this would be incorrect*.

Test your understanding 7

1 State the equation of a linear supply function, and explain the meaning of the two variables and the two parameters.

2 Explain the meaning of 'slope', and define it for the case where the dependent variable is plotted on the horizontal axis.

3 Explain the significance of a positive slope.

4 Given the data in the following table, find the corresponding linear function (equation).

Q_s (dependent variable)	P (independent variable)
50	0
100	25
150	50
200	75
250	100
300	125
350	150

5 Using the equation you found in question 4, **(a)** find the values of Q_s when $P = 17, 25, 37$, and **(b)** find the values of P when $Q_s = 60, 75, 80$.

6 Using the equation you found in question 4, graph the corresponding curve by finding points that do not appear in the table of question 4, following the economics practice of putting the dependent variable on the horizontal axis.

7 **(a)** State the horizontal intercept of the curve that corresponds to the equation of question 4, and identify it on your graph. **(b)** What parameter in a supply equation does the horizontal intercept correspond to?

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The demand function

Understanding the equation of a demand function

Demand, you may remember, involves a negative causal relationship between price and quantity demanded. The data in Table 9 show a negative, linear relationship between price, P, in \$, and quantity demanded, Q_d, in thousand units of good Alpha demanded per day. These data are graphed in Figure 17(a).

Table 9 Data for a demand curve: negative linear relationship

Quantity demanded (Q_d, in thousand units of Alpha per day) (dependent variable)	Price (P, in \$) (independent variable)	Point on graph (Figure 17(a))
100	0	A
80	10	B
60	20	C
40	30	D
20	40	E
0	50	F

The equation of a demand function (negative linear function)

The equation of a demand function is given by:

$$Q_d = a - bP$$

where

Q_d = the dependent variable

P = the independent variable

a = the value of the dependent variable when the independent variable, P, is equal to zero

–b = the slope (where $\text{slope} = \frac{\Delta Q_d}{\Delta P}$), and is the coefficient of the independent variable, P.

You can see that both variables, Q_d and P, and both parameters, a and –b, have the same interpretation as in a supply function, or a positive linear function; the demand function differs only in that the slope has a negative sign. This means there is a negative relation between the two variables, Q_d and P, shown also in Table 9.

Specifying the demand function

Finding the parameters a and –b will allow us to specify the equation describing this relationship of Table 9. Since a represents the value of Q_d when P = 0, we can see immediately in Table 9 that a = 100.

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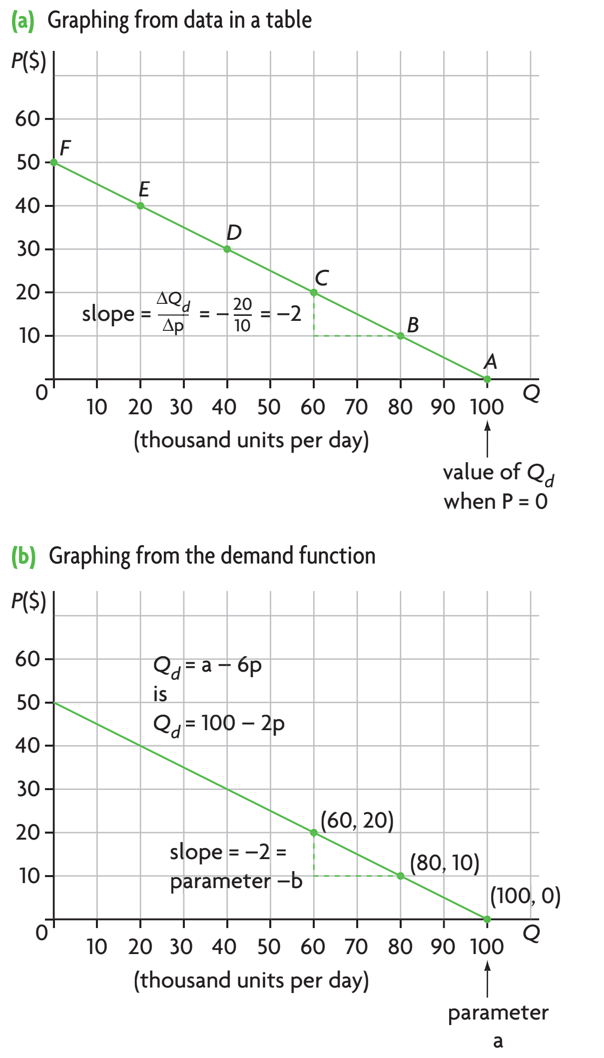

Figure 17 Graphing the demand curve

We know that –b = the slope, where the slope = $\frac{\Delta Q_d}{\Delta P}$; taking points B and C in Table 9, we find:

$\Delta Q_d = 60 - 80 = -20$, and $\Delta P = 20 - 10 = 10$. Therefore,

$$\text{slope} = \frac{\Delta Q_d}{\Delta P} = \frac{-20}{10} = -2$$

(Remember the point made earlier about the need to be consistent in subtracting values of variables.) The negative sign of the slope indicates the negative (indirect) relationship between variables *P* and Q_d.

The equation we are looking for is therefore:

$$Q_d = 100 - 2P$$

Using this equation we can find any value of Q_d, given *P*. For example, if *P* = 30,

$$Q_d = 100 - 2(30) = 40$$

Similarly, we can find any value of *P*, given a value of Q_d. If $Q_d = 25$,

$$25 = 100 - 2P \quad \Rightarrow \quad 2P = 75 \quad \Rightarrow \quad P = 37.5$$

The equation of a linear demand function is given by $Q_d = a - bP$, where Q_d is quantity demanded (the dependent variable), *P* is price (the independent variable), a is the value of Q_d when *P* = 0, and –b is the slope (given by $\frac{\Delta Q_d}{\Delta P}$).

The negative sign of the slope (–b) indicates that the demand function represents a negative (indirect) relationship between *P* and Q_d.

Graphing the demand curve

To plot the curve that corresponds to the equation $Q_d = 100 - 2P$ (assuming we do not have the data in Table 9), we must find at least two points on the line. We already have one, which is the parameter a = 100, meaning that $Q_d = 100$ when *P* = 0, i.e. point (100,0). To get a second point, we can let *P* = 10, in which case $Q_d = 100 - 2(10) = 100 - 20 = 80$, which is point (80,10). For a third point, if *P* = 20, $Q_d = 100 - 2(20) = 60$, or point (60,20). We can now plot the three points.

The points are plotted in Figure 17(b). Note that as in the case of the supply curve, we obtain *the demand curve by plotting the dependent variable on the horizontal axis and the independent variable on the vertical axis.*

We can now make a similar point as in the case of the supply function:

When the dependent variable appears on the horizontal axis, in the equation of a demand function of the form $Q_d = a - bP$, the parameter a is equal to the horizontal intercept, and the parameter –b, or slope, is equal to the horizontal change divided by the vertical change.

An important note on the slope

The point made earlier (page 19) concerning the practice of reversing the axes in supply functions applies equally to demand functions. The slope we are calculating by plotting the dependent variable on the horizontal axis is different than if we had followed the correct mathematical practice.

HL

Test your understanding 8

1 State the equation of a linear demand function, and explain the meaning of the two variables and the two parameters.

2 Explain the significance of the negative slope of the demand function.

3 Given the data in the accompanying table, find the corresponding linear function (equation).

Q_d (dependent variable)	P (independent variable)
50	0
45	2
40	4
35	6
30	8
25	10
20	12
15	14
10	16

4 Using the equation you found in question 3, (a) find the values of Q_d when $P = 3$, $P = 4.2$, $P = 11$; (b) find the values of P when $Q_d = 27.5$, $Q_d = 35$, $Q_d = 42.5$.

5 Using the equation you found in question 3, graph the demand curve, by finding points that do not appear in the table in question 3, following the economics practice of putting the dependent variable on the horizontal axis.

6 Using your equation from question 3, calculate the vertical and horizontal intercepts, and identify them in your graph of question 5.

7 What parameter in a demand equation does the horizontal intercept correspond to?

8 For each of the following equations, state whether they describe a positive or negative relationship, and explain why.

(a) $Q = -5 + 7P$

(b) $Q = 17 - 5P$

(c) $Q = 10 + 15P$

Graphing the relevant parts of demand and supply curves

When we plot supply and demand curves, we are interested only in those portions of the curves that appear in quadrant I (defined on page 8), i.e. that have only positive values for both the Q and P variables (it is not possible to have negative prices or quantities). In Figure18, the bold-face portions of the curves indicate the relevant portions of supply curves in part (a), and the relevant portions of demand curves in part (b).

In the general supply function, $Q_s = c + dP$, we know that the parameter c represents the horizontal intercept. Looking at Figure 18 (a), we can see that c can have a negative value, as in S_1, or a positive value, as in S_2. This means that glancing at a supply function with numerical values for c, we can see if the corresponding curve is of the form S_1 or S_2. For example, a function $Q_s = -20 + 10P$ is of the form S_1; $Q_s = 20 + 5P$ is of the form S_2.

When the parameter c has a negative value, we are not interested in calculating c (the horizontal intercept) as this represents a range of the curve where Q_s is negative, and we do not want to plot negative Q_s

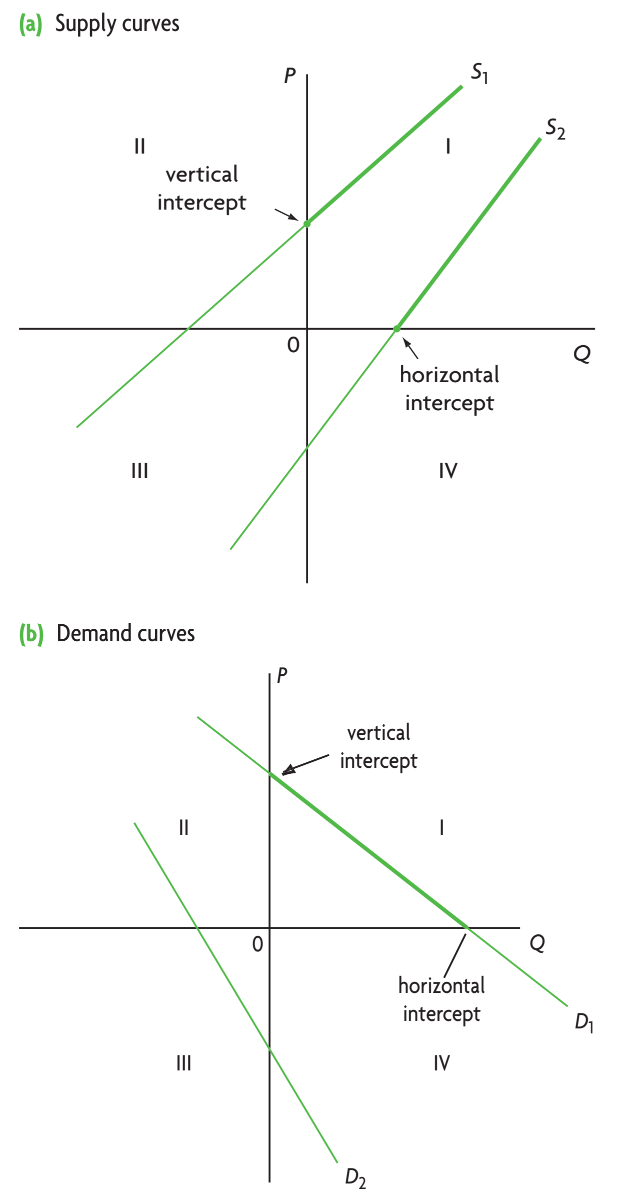

Figure 18 The relevant portions of supply and demand curves

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values. Instead, finding the vertical intercept is more useful, as this is the point where the supply curve begins. For example, suppose S_1 in Figure 18(a) is given by $Q_s = -20 + 10P$ (graphed in Figure 19). We know the vertical intercept is where $Q = 0$; therefore, setting $Q = 0$ in the equation we solve for P:

$$Q = 0 = -20 + 10P \quad \Rightarrow \quad 20 = 10P \quad \Rightarrow \quad P = 2$$

giving the vertical intercept at point (0,2). We can therefore plot the solid portion of the curve, for positive values only.

In the case of S_2 in Figure 18(a), the parameter c has a positive value (in this range Q_s is positive); therefore, the horizontal intercept represents the beginning of the supply curve. Given a function $Q = 20 + 5P$, graphed in Figure 19, we set $P = 0$ (as we have done in other examples above), and find $Q = 20$; therefore, the horizontal intercept is at the point (20,0).

In Figure 18(b), the demand curve D_2 is meaningless as all P and Q values are negative. All other demand curves take the form of D_1, where the vertical and horizontal intercepts define the endpoints of the demand curve.

In some cases, you may be asked to graph a demand or supply curve within a particular price range. For example, if you are asked to graph the supply function $Q_s = 20 + 5P$ within a price range of $P = 0$ to $P = 4$, this means simply that the supply curve should begin at the price of 0 and end at the price of 4. You can simply calculate the value of Q_s for $P = 0$, which is $Q_s = 20$, or point (20,0) and the value of Q_s for $P = 4$, which is $Q_s = 40$, or point (40,4), and plot these two points as the end points of the supply curve. You will then have a supply curve like the unbroken part of S_2 in Figure 19.

(We have not considered vertical and horizontal curves, which are special cases; you will learn about these in the textbook.)

A note on supply and demand curves

We have seen that the parameter c can take on positive or negative values. If c has a negative value, the supply curve begins somewhere on the P axis, which means that the firm would only be willing to begin supplying its product at a price which is greater than zero; this is shown by S_1 in Figure 18(a). However, what if c has a positive value? A positive c means that $Q_s > 0$ when $P = 0$, or that the firm would be willing to begin supplying its product when the price of the product is zero; this is shown by S_2 in Figure 18(a). Yet how is this possible; what firm would be willing to produce and 'sell' a good if it cannot sell at a price that is greater than zero?

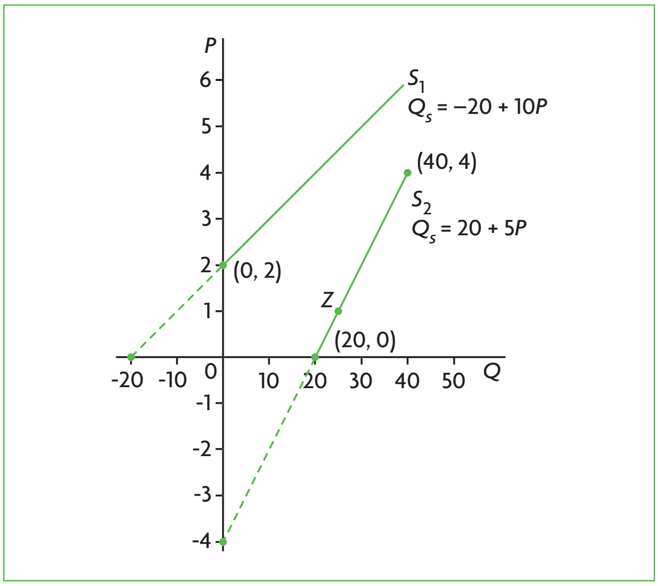

Figure 19 Examples of vertical and horizontal intercepts in supply functions

The answer to this question is that supply curves that have a positive c do not actually begin at a point on the horizontal intercept; they have a lower limit, which is at a price that is greater than zero. This could be at a point such as z on S_2 in Figure 19. The point z represents the lowest price that a firm would accept in order to produce and sell its good. This is a price that allows the firm to cover at least some of its costs. You will learn about this price in Chapter 7 of the textbook.[2]

Therefore, graphing the supply curve all the way to the horizontal axis is a mathematical convenience that we can use when we are not given a lower limit for price. Similarly, plotting a demand curve all the way to the horizontal axis is also a mathematical convenience, since at a price of zero consumers are likely to demand much more than the Q given by the horizontal intercept.

Test your understanding 9

Plot each of the following equations, putting P (in \$) on the vertical axis, and graphing only the relevant portions of the curves.

(a) $Q_s = -20 + 10P$, up to price $P = 7$.

(b) $Q_s = 10 + 15P$, from the horizontal intercept to price $P = 4$.

(c) $Q_d = 15 - 5P$, from the vertical intercept to the horizontal intercept.

(d) $Q_d = 10 - 2P$, from $P = 1$ to $P = 4$.

[2] As you will learn in Chapter 7, this price is the 'shut-down' price, which for a perfectly competitive firm is equal to minimum average variable cost in the short run, and is equal to minimum average total cost in the long run.

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Shifts in demand and supply curves

Rightward and leftward shifts in curves: finding the new equation

The parameters a (in the demand function $Q_d = a - bP$) and c (in the supply function $Q_s = c + dP$) have a very important meaning: they represent all the variables that are held constant under the *ceteris paribus* assumption in the relationship between Q_s or Q_d and P. We now want to see what happens to a linear curve when the parameter a or c changes.

We know from our earlier discussion (pages 11 and 12) that any change in a determinant of a functional relation, in other words, any change in one of the factors held constant by use of the *ceteris paribus* assumption, leads to a shift of the curve. This is exactly what happens if there is a change in the parameters a or c.

Shifts in the demand function

Figure 20(a) shows shifts that occur when there is a change in the parameter a in the demand function $Q_d = a - bP$. In the function $Q_d = 100 - 2P$, representing consumers' daily demand for Alpha (see Table 9), parameter a = 100, meaning that consumers would want 100 thousand units of the good each day if $P = 0$. Suppose then that consumers' income increases, so that at each price, they want to buy 10 thousand additional units of the good. This means that when $P = 0$, consumers will want to buy 100 + 10 = 110, actually 110 thousand units; therefore, a increases from a = 100 to a = 110. This gives rise to a new equation:

$$Q_d = 110 - 2P$$

The new equation is graphed in Figure 20(a), and results in a parallel rightward shift of the demand curve by 10 units at every price, or actually 10 thousand units (which are measured along the horizontal axis). The new Q intercept is at $Q = 110$, which is 10 units to the right of the old Q intercept of $Q = 100$.

On the other hand, suppose that a fall in income leads to a decrease in consumer demand of 20 thousand units at each price (relative to the initial demand curve). This means that when $P = 0$, consumers will want to buy 100 − 20 = 80, i.e. 80 thousand units; therefore, a falls from a = 100 to a = 80. This results in the equation:

$$Q_d = 80 - 2P$$

In Figure 20(a), the graph of this equation shows a parallel leftward shift of the demand curve by 20 units at every price, or actually 20 thousand units (measured along the horizontal axis). The new Q intercept is at $Q = 80$, which is 20 units to the left of the initial Q intercept of $Q = 100$.

Shifts in the supply function

In the supply function $Q_s = 5 + \frac{1}{4}P$, representing a firm's supply curve for good Alpha (see Table 8), c = 5, which means that when $P = 0$, the quantity supplied would be 5 thousand units per day. (You may want to bear in mind the point made earlier, page 23, that a supply curve that begins at the Q intercept when $c > 0$ is a mathematical convenience, since no firm would supply its product at a price of zero.) Suppose there is a fall in costs of production, resulting in an increase in supply of 10 thousand units per day at every price. This means that when $P = 0$, the quantity supplied would be 5 + 10 = 15, i.e. 15 thousand units per day of Alpha.

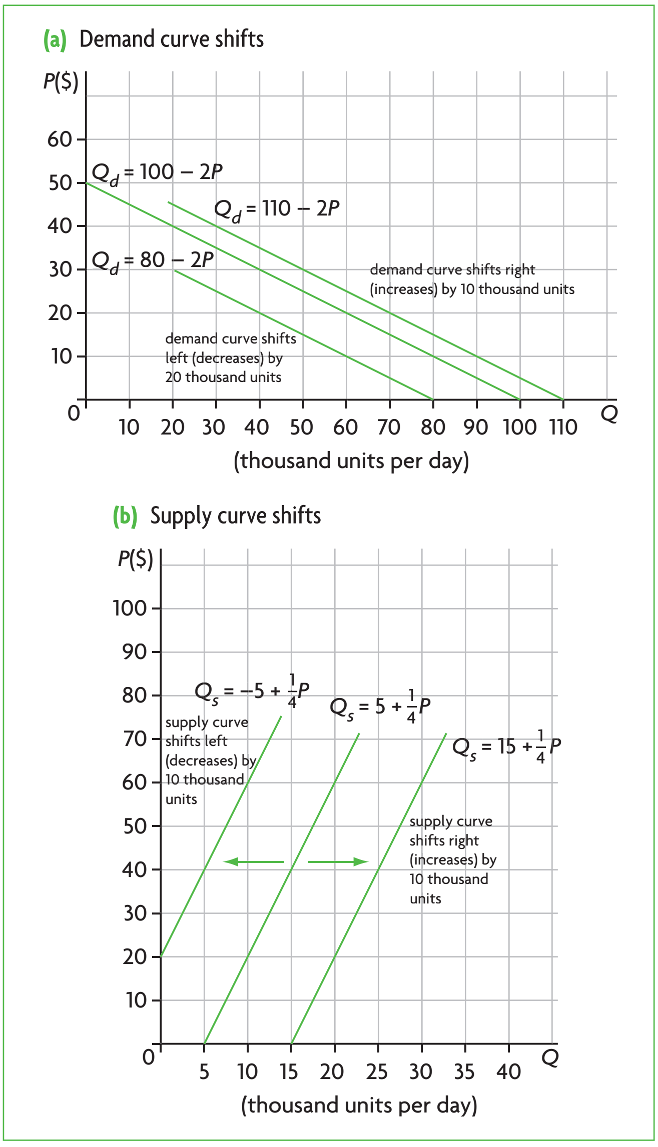

Figure 20 Shifts in linear demand and supply curves

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Therefore, the parameter c increases from c = 5 to c = 15, and the new equation becomes:

$$Q_s = 15 + \frac{1}{4}P$$

When the above equation is graphed, there emerges a new linear curve parallel to the initial one, beginning at the point where $Q = 15$, occurring when $P = 0$, or point (15,0), as shown in Figure 20(b). Since the parameter c increased from 5 to 15, this means that the curve has shifted rightward by the amount of 10 units of Q (= 15 − 5); note that this is actually 10 thousand units of Q, since Q is measured in thousands of units. Therefore, for each value of P, the entire new curve lies 10 thousand units of Q to the right of the initial curve.

Suppose, instead, that relative to the initial function $Q_s = 5 + \frac{1}{4}P$, there is an increase in costs of production, resulting in a decrease in daily supply of 10 thousand units at every price. This means that when $P = 0$, $Q_s = -5$ units. The parameter c decreases from 5 to −5, and the new equation becomes:

$$Q_s = -5 + \frac{1}{4}P$$

In this case, the supply curve shifts to the left by 10 units of Q (actually 10 thousand units) at every price. The new curve can be graphed by finding points that lie 10 units to the left of the initial supply curve. Note that c = −5 means that $Q_s = -5$ when $P = 0$, which is in the negative range of Q. In fact, the supply curve begins at $Q = 0$, where $P = 20$ (or the vertical intercept), found by setting $Q = 0$ and solving:

$$0 = -5 + \frac{1}{4}P \quad \Rightarrow \quad 5 = \frac{1}{4}P \quad \Rightarrow \quad P = 20$$

Note that in the case of all changes in the parameters a and c, the slope of the curves remained the same, explaining why the new curves are in all cases parallel to the initial ones.

In curves described by the linear functions, $Q_d = a - bP$ or $Q_s = c + dP$, a change in parameters a or c indicates a change in a variable that was held constant under the *ceteris paribus* assumption, and causes a parallel shift of the curve. If a or c increases, there is a rightward shift by the amount of the increase (measured along the horizontal axis); if a or c decreases, there is a leftward shift by the amount of the decrease (measured along the horizontal axis).

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Test your understanding 10

1 Using your graphs from the questions in Test your understanding 9, show (by graphing) what will happen to each curve if:

(a) in the function $Q_s = -20 + 10P$, there is an increase in supply of 10 units at every price

(b) in the function $Q_s = 10 + 15P$ there is a decrease in supply of 30 units at every price

(c) in the function $Q_d = 15 - 5P$ there is a decrease in demand of 5 units at every price

(d) in the function $Q_d = 10 - 2P$ there is an increase in demand of 2 units at every price.

2 State the new equation for each of the changes in question 1.

Upward and downward shifts in curves: finding the new curve and equation

On pages 11 and 13 above, it was noted that rightward/leftward shifts of curves can also be viewed as upward/downward shifts. This is especially relevant to supply curves, which in some cases are considered as shifting to the right or to the left (Chapter 2) and other times upward or downward (Chapters 4 and 5).

It is convenient to consider supply curve shifts as going in the up/down direction in the study of certain kinds of taxes called indirect taxes, as well as subsidies, as both of these have the effect of changing firms' costs of production. (Remember that subsidies are payments by the government to firms, and are therefore the opposite of taxes.) The reason is that *the vertical difference between the old and new supply curves represents the change in the firm's costs of production due to the indirect tax or subsidy*.

The case of indirect taxes

Let's consider the following example. We begin with the supply function $Q_s = 5 + \frac{1}{4}P$, which is the same as the function considered above (page 19). Suppose the government imposes an indirect tax on good Alpha produced by firms of \$40 per unit of Alpha sold. This will cause the supply curve to *shift upward by the amount of \$40 for each quantity of Alpha sold*. The pre-tax and after-tax supply curves are shown in Figure 21(a), with the vertical difference between S_1 and S_2 representing the *tax per unit*.

We would now like to find the equation of the new supply curve. To do this, we use the following rule. Given a supply function of the general form

[3] You may be surprised to see that we subtract t from P when we are shifting the supply curve upward. The reason is that subtracting t from P actually means we are shifting the axes downward, thus actually moving the supply curve upward.

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$Q_s = c + dP$, whenever there is an *upward shift* of the function by t units, we replace P by $P - t$.[3] The new supply function therefore becomes:

$$Q_s = c + d(P - t)$$

In our case, t = 40. Therefore, given our pre-tax equation of $Q_s = 5 + \frac{1}{4}P$, the new, after-tax equation is

$$Q_s = 5 + \frac{1}{4}(P - 40) \quad \Rightarrow \quad Q_s = 5 + \frac{1}{4}P - 10$$

$$Q_s = -5 + \frac{1}{4}P$$

You may now notice something interesting. The after-tax equation we have derived is identical to the equation we found earlier (page 25), when the curve of the equation $Q_s = 5 + \frac{1}{4}P$ shifted to the left due to a decrease in supply of 10 thousand units because of an increase in costs of production. This is not surprising, because in fact the two shifts are identical. If you examine Figure 20(b) and Figure 21(a), you will see that the two curves are also identical. They have just been derived differently. Table 10 summarises the two ways of deriving the new supply curve and equation.

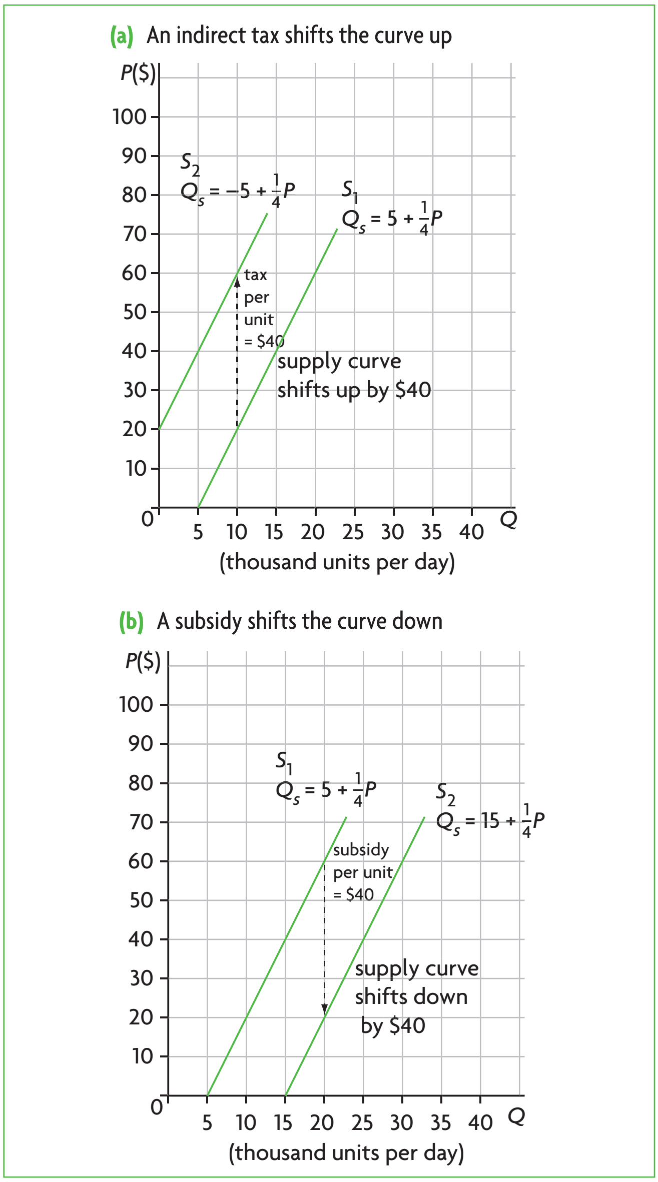

Figure 21 Upward/downward shifts of the supply curve

Table 10 Leftward shift and upward shift in the supply curve

Initial equation: $Q_s = 5 + \frac{1}{4}P$

Decrease in supply: leftward shift	Indirect tax: upward shift
Increase in production costs causes a decrease in supply of 10 units at every price.	A tax of $40 per unit causes an increase in production costs for every quantity supplied.
Supply curve shifts *leftward* by 10 units measured along the horizontal axis.	Supply curve shifts *upward* by $40 measured along the vertical axis.
To find the new supply function: find the new c, which is c – 10 = 5 – 10 = –5, and rewrite supply function using parameter c = –5.	To find the new supply function: rewrite supply function as $Q_s = 5 + \frac{1}{4}(P - 40)$ and simplify.

Final equation: $Q_s = -5 + \frac{1}{4}P$

The case of subsidies

We can now consider the case of a subsidy, which results in a decrease in the firm's costs of production. We begin with the same initial supply function as before, $Q_s = 5 + \frac{1}{4}P$, and suppose that the government grants a subsidy to firms of $40 per unit of Alpha sold. This will cause the supply curve *to shift downward by the amount of $40 for each quantity of Alpha sold*. The pre-subsidy and after-subsidy supply curves are shown in Figure 21(b), with the vertical difference between S_1 and S_2 representing the subsidy per unit.

To find the equation of the new supply curve, we use the following rule. Given a supply function of the general form $Q_s = c + dP$, whenever there is a *downward shift* of the function by s units, we replace P by $P + s$.[4] The new supply function therefore becomes:

$$Q_s = c + d(P + s)$$

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[4] Note that this is the exact opposite of what we did to find the new supply function when a tax was imposed (page 25). We now add s to P in order to shift the supply curve downward, because by doing so we are shifting the axes upward, thus actually moving the supply curve downward.

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Since in our case s = $40, our equation of $Q_s = 5 + \frac{1}{4}P$, after the subsidy becomes:

$$Q_s = 5+\frac{1}{4}(P+40) \Rightarrow Q_s = 5+\frac{1}{4}P+10$$
$$Q_s = 15+\frac{1}{4}P.$$

The equation (after the subsidy) we have just derived is the same as the equation we found when there is an increase in supply of 10 thousand units due to a fall in costs of production. Again, this has occurred because the two shifts are identical. The two resulting supply curves are also identical, as you can see by comparing Figure 20(b) with Figure 21(b). Table 11 summarises the results.

Table 11 Rightward shift and downward shift in the supply curve

Initial equation: $Q_S = 5 + \frac{1}{4}P$

Increase in supply: rightward shift	Subsidy: downward shift
Decrease in production costs causes an increase in supply of 10 units at every price.	A subsidy of $40 per unit causes a decrease in production costs for every quantity supplied.
Supply curve shifts *rightward* by 10 units measured along the horizontal axis.	Supply curve shifts *downward* by $40 measured along the vertical axis.
To find the new supply function: find the new c, which is c + 10 = 5 + 10 = 15, and rewrite supply function using parameter c = 15.	To find the new supply function: rewrite supply function as $Q_s = 5 + ¼\,(P + 40)$ and simplify.

Final equation: $Q_S = 15 + \frac{1}{4}P$

If you are given a supply function and are asked to graph the corresponding supply curve, and then to graph a *new supply curve* following a change of some kind, *you must be very careful to consider the nature of the change*. If it involves a change in supply, you must shift the supply curve leftward or rightward by the amount of the change in supply. If it involves an indirect tax or subsidy, you must use the information to shift the supply curve upward or downward by the amount of the tax or subsidy per unit of output.

All changes in supply (increases or decreases) are analysed as rightward or leftward shifts of the supply curve. Indirect taxes and subsidies are analysed as upward and downward shifts of the supply curve because of their effect on firms' costs of production.

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Test your understanding 11

1 Using your graphs from the questions in Test your understanding 9, show (by graphing) what will happen to the supply curves if:

(a) in the supply curve given by $Qs = -20+10P$, the government grants a subsidy of $1 per unit of the good; show the per unit subsidy in your diagram;

(b) in the supply curve given $Qs = 10+15P$, the government imposes a tax of $2 per unit of the good; show the per unit tax in your diagram.

2 Find the new supply equations that result after the changes described in question 1 occur.

3 Compare the graphs you drew in question 1 above with the graphs you drew for Test your understanding 10, question 1 parts (a) and (b), and explain why they are the same. Are the corresponding equations also the same? (They should be.)

The slope

Interpreting changes in parameters –b and d (the slope)

When the slope of a linear demand function (–b) or linear supply function (d) changes, the steepness of the curve changes. Suppose that we have a supply function, $Q_s = 10 + 4P$, shown as S_1 in Figure 22(a). If the slope changes from 4 to 5, the new equation becomes:

$$Q_s = 10 + 5P$$

also shown in Figure 22(a). Note that the larger value of the slope (5 > 4) has made the curve flatter.

A demand function, $Q_d = 50-5P$, is shown in Figure 22(b). If the slope changes from –5 to –10, the new equation becomes:

$$Q_d = 50 - 10P$$

which can also be seen in Figure 22(b). Note that the larger *absolute value* of the slope has made the curve flatter (the absolute value is the numerical value without the minus sign. Therefore, the absolute value of –10 is 10).

(When the dependent variable is plotted on the vertical axis (as in the usual mathematical practice), the opposite holds: the larger the absolute value of the slope, the steeper the line.)

HL

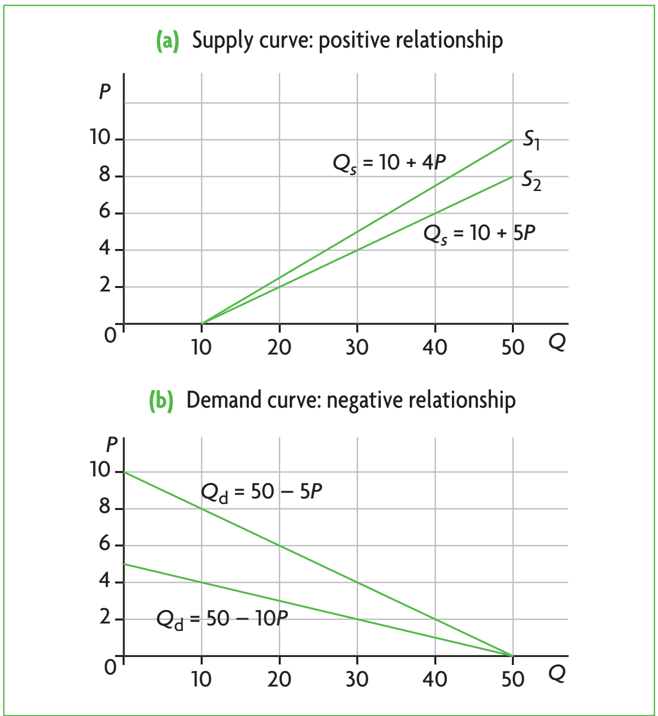

Figure 22 Changes in the slope of linear curves (with the dependent variable on the horizontal axis)

In a linear demand or supply function, $Q_d = a - bP$ or $Q_s = c + dP$, a change in the slope (–b or d) results in a change in the steepness of the curve. When the dependent variable is plotted on the horizontal axis, the larger the absolute value of the slope, the flatter the line.

Test your understanding 12

1 State the new equation for each of the following changes: **(a)** in the function $Q_s = -20+10P$, the slope changes to +25, and **(b)** in the function $Q_d = 15-5P$, the slope changes to –10.

2 For each of the following linear equations where the dependent variable Q, is plotted on the horizontal axis, state whether the new curve will become flatter or steeper. (a) in $Q_s = 5+20P$ the slope changes to 15, (b) in $Q_d = 4-12P$ the slope changes to –8, (c) in $Q_s = -3+7P$ the slope changes to 9, and (d) in $Q_d = 2-15P$ the slope changes to –17.

It should be noted that the slope of a straight line is always constant, meaning that it is the same no matter on what part of the line it is measured. The same is not true of non-linear curves, whose slope is continuously changing.

Comparing slopes of linear curves in different diagrams

It was stated above that the greater the absolute value of the slope, the flatter the curve if the dependent variable is on the horizontal axis, and the steeper the curve if the dependent variable is on the vertical axis. You should be careful to note, however, that these points apply only to curves that are drawn in the same diagram that uses the same units, or that are drawn in the same scale. If two curves are in different diagrams with different scales, it is not possible to compare the slopes by reference to the steepness of the linear curves.

Slope and elasticity

The slope, defined as $\frac{\Delta Q}{\Delta P}$, measures the responsiveness of the dependent variable to changes in the independent variable. You can see this in Figure 22. In part (a), suppose there is a change in P (the independent variable) from 4 to 5. What will be the effects on the dependent variable Q_s? In the supply curve defined by $Q_s = 10+4P$, when $P = 4$, $Q_s = 26$; when $P = 5$, $Q_s = 30$. Therefore, Q_s increases by 4 units. In the *flatter* supply curve defined by $Q_s = 10+5P$, when $P = 4$, $Q_s = 30$; when $P = 5$, $Q_s = 35$, i.e. it increases by 5 units. Therefore, *there is a larger responsiveness in* Q_s *when the curve is flatter.*

The idea of responsiveness of one variable to changes in another variable is very important in economics, and we will encounter it repeatedly throughout this course. Therefore, the slope is an important concept. However, as a measure of responsiveness it suffers from a serious limitation: it depends on the particular units used to measure the variables. If units change, the slope also changes; the use of different units means that responsiveness is not comparable by use of the slope.

Economists therefore use a closely related concept to measure responsiveness, called **elasticity**. Whereas *slope measures responsiveness in absolute terms, elasticity measures responsiveness in percentage terms:*

$$\text{slope} = \frac{\Delta Q}{\Delta P} \qquad \text{elasticity} = \frac{\%\Delta Q}{\%\Delta P} = \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}}$$

$$= \frac{\Delta Q}{\Delta P} \times \frac{P}{Q} = \text{slope} \times \frac{P}{Q}$$

The concept of elasticity will be studied in Chapter 3 of the textbook.

HL

Solving simultaneous linear equations to calculate equilibrium price and quantity

Suppose we are given a linear demand function $Q_d = 45 - 5P$, and a linear supply function $Q_s = -30 + 20P$. The two curves defined by the two functions have a point of intersection, or a point where they cross each other, which we would like to find as this has a major economic significance. We can find the point of intersection by graphing the two curves, as in Figure 23, which shows that the two curves cross each other at the point where $P = 3$ and $Q = 30$, or the point (30,3).

However, we would like to find the point of intersection mathematically, because this gives far more accurate results. To do this, we must solve for P and Q using the equations of the demand and supply functions. This is done in the following way.

We have the following two equations:

$$Q_d = 45 - 5P \text{ (the demand function)}$$
$$Q_s = -30 + 20P \text{ (the supply function)}$$

In addition, we know that $Q_d = Q_s$ at the point where the two curves cross, since at that point quantity demanded is equal to quantity supplied. Therefore, we can set the demand and supply functions equal to each other:

$Q_d = Q_s$ can be written as

$$45 - 5P = -30 + 20P$$

Solving for P, we have:

$$45 + 30 = 5P + 20P \quad \Rightarrow \quad 75 = 25P$$
$$P = \frac{75}{25} = 3$$

We now need to find Q. To do this, we substitute $P = 3$ into either the demand or supply function above, and solve for Q:

$$Q_d = 45 - 5(3) = 45 - 15 = 30$$

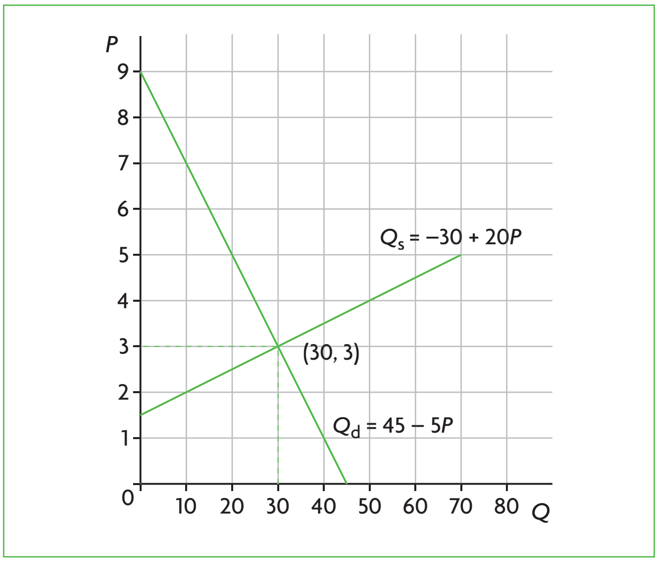

Figure 23 Demand and supply curves and their point of intersection

To check our results we could substitute $P = 3$ into the supply equation as well:

$$Q_s = -30 + 20(3) = -30 + 60 = 30$$

Since both methods give the same result, we can be fairly confident that our calculations are correct.

Therefore, the point of intersection is (30,3), or where $P = 3$ and $Q = 30$. As we will discover in Chapter 2, this price and quantity combination is known as **equilibrium price and quantity**.

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Test your understanding 13

1 You are given the following demand function: $Q_d = 10 - 2P$, where Q_d is quantity of good Beta demanded in thousands of units per week, and P is the price of Beta in \$. **(a)** Identify the horizontal intercept. **(b)** Calculate the vertical intercept. **(c)** Identify the slope. **(d)** Graph the curve that corresponds to this function. **(e)** Explain whether this is a positive or negative relationship; how does your answer relate to the slope? **(f)** Assume the parameter 10 increases to 14; show this graphically (up to a price $P = 5$) and explain in words the meaning of this shift. **(g)** State the new equation. **(h)** Starting with the initial demand function, assume the slope changes to –1; state the new demand equation and explain what will happen to the steepness of the demand curve.

2 You are given the following supply function: $Q_s = -3 + 3P$, where Q_s is quantity of good Beta supplied in thousands of units per week, and P is the price of Beta in \$. **(a)** Identify the horizontal intercept; should this be included in a graph of the supply curve? **(b)** Calculate the vertical intercept. **(c)** Identify the slope. **(d)** Graph the curve that corresponds to this function from $P = 1$ to $P = 4$. **(e)** Explain whether this is a positive or negative relationship; how does your answer relate to the slope? **(f)** Assume that the parameter –3 changes to –1; show this graphically and explain in words the meaning of this shift. **(g)** State the new equation. **(h)** Assume the slope changes to 2; state the new supply equation and explain what will happen to the steepness of the supply curve.

3 **(a)** Using the demand function, $Q_d = 10 - 2P$, and the supply function, $Q_s = -3 + 3P$, where Q_d and Q_s refer to thousands of units of Beta per week and P is in \$, solve for P and Q and determine equilibrium price and quantity. **(b)** Plot the two curves on the same graph from $P = 1$ to $P = 4$; is their point of intersection the same as the one you calculated for P and Q?

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4 **(a)** Calculate Q_d and Q_s that would result at a price of \$4. **(b)** What is the excess (or extra) quantity of Beta supplied at $P = \$4$?

5 **(a)** Calculate Q_d and Q_s that would result at a price of \$2. **(b)** What is the excess (or extra) quantity of Beta demanded at $P = \$2$?

6 Suppose there is an increase in demand for Beta of 2000 units at each price. **(a)** Find the new demand equation. **(b)** Find the new equilibrium price and quantity by solving the equations. **(c)** Graph the new demand curve in your diagram for question 3 above and identify the equilibrium price and quantity. Do they match with your calculations?

7 Solve the following equations for P and Q: $Q_s = 10 + 5P$ and $Q_d = 50 - 5P$.

Using a graphics display calculator (GDC) to graph functions and find equilibrium price and quantity

You are permitted to use a graphics display calculator (GDC) when taking a higher level paper 3 exam. A GDC can be used mainly to help you graph functions. However, *you should note that a GDC is not essential to answering paper 3 questions*. This section is intended to guide you through how you can use a GDC for higher level paper 3, should you decide you would like to do so.

Graphing a demand function using a GDC

A GDC expresses functions in the following way:

$$y = f(x)$$

which means that the variable y is a function of x, and *where* y *is always plotted on the vertical axis, and* x *is always plotted on the horizontal axis*.

Suppose you are asked to graph the following demand function:

$$Q_d = 6 - 2P$$

The Q intercept is $Q = 6$, and the P intercept is $P = 3$ (obtained by setting $Q_d = 0 = 6 - 2P \Rightarrow 2P = 6 \Rightarrow P = 3$). This demand curve is graphed in Figure 24(a).

Suppose now you wanted to use your GDC to graph $Q_d = 6 - 2P$. If you were to write the demand function as

$$y = 6 - 2x$$

where you have set $y = Q_d$ and $x = P$, and you put this into your calculator to get its graph, the graph that will appear on your screen will look like the graph in Figure 24(b). If you compare the curve in part (a)

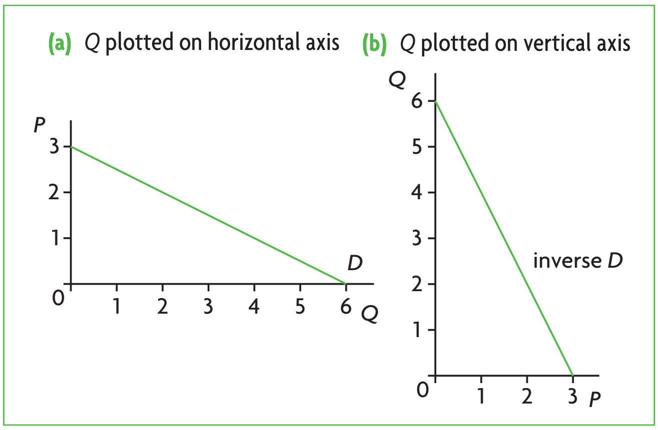

Figure 24 Demand curves $Q_d = 6 - 2P$

with the curve in part (b), you will see that they are different. In fact, *the curve that appears in Figure 24(b) is incorrect from an economics perspective, because it plots* Q *on the vertical axis and* P *on the horizontal axis*.

In order to find the correct graph of the demand curve on your calculator, the demand function $Q_d = 6 - 2P$ has to be expressed in a way so that when the calculator graphs it, Q_d will appear on the horizontal axis and P on the vertical axis. In other words, the variables in the function $y = f(x)$, must be identified as $y = P$ (plotted on the vertical axis) and $x = Q_d$ (plotted on the horizontal axis. For this to happen, the demand function $Qd = 6 - 2P$ must be rewritten so that P is expressed as a function of Q. This involves taking the initial demand function and solving for P:

$$Q_d = 6 - 2P \quad \Rightarrow \quad 2P = 6 - Q_d \quad \Rightarrow \quad P = 3 - \frac{1}{2}Q_d$$

The equation $P = 3 - \frac{1}{2}Q_d$, where P is expressed as a function of Q, is called the *inverse demand function*. In graphing the inverse demand function, the GDC will provide the correct demand curve, as in Figure 24(a). The reason is that now the calculator has correctly identified P as the variable y that is plotted on the vertical axis, and Q_d as the variable$\times$that is plotted on the horizontal axis.

In practice, you do not have to calculate the inverse demand function yourself, as the GDC can do it for you, and then graph it. To get the correct graph, enter the initial demand function $Q_d = 6 - 2P$ into the calculator, and ask it to graph *the inverse function* (the steps for this are described at the end of this chapter). This will provide you with the correct graph of the demand curve.

Graphing a supply curve using a GDC

Graphing a supply curve with the help of a GDC is identical to graphing a demand curve. Suppose you

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want to graph the function $Q_s = 2+2P$. This graph appears in Figure 25(a). If your GDC graphs this function, it will come up with what appears in Figure 25(b), which is incorrect because Q has been plotted on the vertical axis and P on the horizontal axis. The GDC will find the correct supply curve by plotting the *inverse supply function*, which involves expressing P as a function of Qs. The inverse supply function in this case would be $P = -1 + \frac{1}{2} Q_s$. However, you need not solve for P yourself to find the inverse function, as the calculator will do it for you. You simply enter the initial supply function into the GDC, and ask it to graph the inverse function.

Given a demand or supply function, to graph the demand curve or supply curve using a GDC, enter the function into the calculator and ask it to graph the *inverse function*. The graph that will appear will be the correct demand or supply curve.

Finding the equilibrium price and quantity

Price and quantity at the point of equilibrium correspond to the value of P and the value of Q where the demand curve and supply curve intersect (cross each other). As we saw earlier, this point of intersection can be found (a) graphically, which involves reading off your graph of the demand and supply curves the values of P and Q at the point where they cross, or (b) mathematically, which involves solving the demand and supply equations for P and Q.

If you are asked to find equilibrium P and Q mathematically, your GDC will not be of any use, as it does not do the necessary manipulations.

If you are asked to find P and Q graphically, this follows simply from your graphs of the demand and supply curves that you have already produced using a GDC to graph the inverse demand function and the inverse supply function. This graph appears in Figure 26(a), which shows that equilibrium $P = 1$ and equilibrium $Q = 4$, and the demand and supply curves are correctly drawn.

The only further use that a GDC can have in the context of finding equilibrium P and Q is to help you check the accuracy of your calculations or of your graphs. This can be done in the following way.

Enter the demand function, $Q_d = 6-2P$, and the supply function, $Q_s = 2+2P$, into the calculator, and ask it to graph them and find the point of intersection. However, you must be very careful here. The calculator will provide you with a graph as in Figure 26(b), which *provides you with the correct values of* P *and* Q*, but the wrong curves*. As you can see in Figure 26(b), *the point of intersection occurs at* Q = 4 *and* P = 1*, which are the same values for* P *and* Q *as in Figure 24(a); however, the curves are incorrect because they plot* Q *on the vertical axis and* P *on the horizontal axis*. Therefore, you can only use this method to check that the values for P and Q you have calculated are correct.

In fact, considering the time constraints in an exam, it may not be worthwhile for you to go to such lengths to check your calculations.[5]

Shifts in the functions

Once you have graphed demand and/or supply curves, you may be asked to find a new demand curve, or a new supply curve, following changes in the parameter a in the function $Q_d = a - b\,P$, or in the parameter c

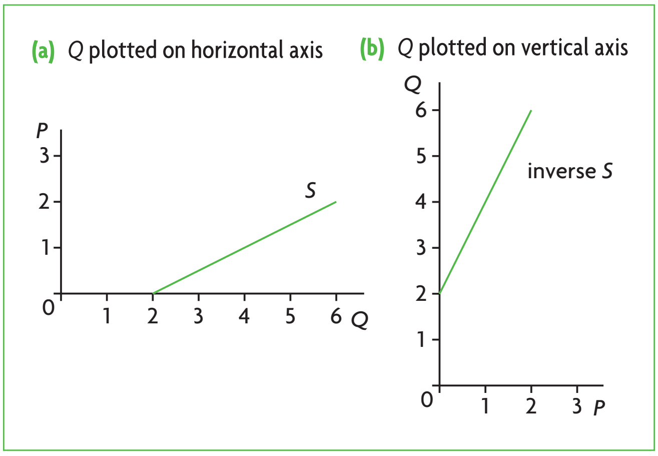

Figure 25 Supply curves $Q_s = 2+2P$

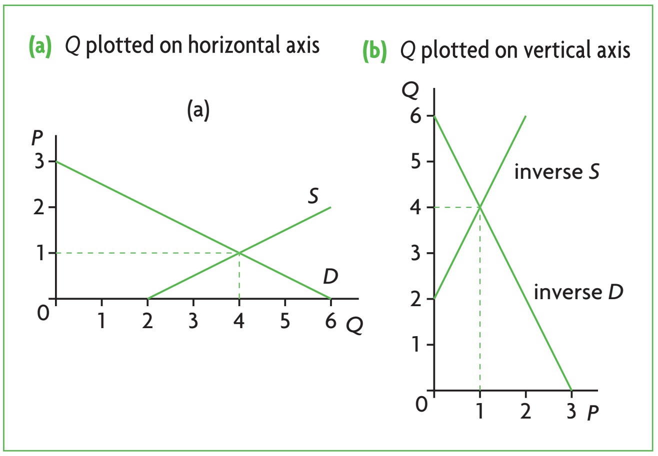

Figure 26 Demand, supply and equilibrium P and Q

[5] It is also possible for you to calculate the inverse demand function and the inverse supply function, by solving in each case for P, enter these new equations into the calculator, and ask it to find the point of intersection. In this case, you will get both the correct values for P and Q, and the correct demand and supply curves. However, this is very time-consuming, and is not recommended.

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in the function $Q_s = c + dP$. You may then be asked to graph the new demand or supply curve.

A GDC cannot help you here, *unless you are able to first find the new equation* that results following a change in the parameter a or c. The method for finding such new equations was explained in detail above (pages 24 and 25). Once you have found a new equation, enter it into the calculator and ask it to graph the inverse function. This will give you the graph of the new demand curve or new supply curve that you require.

For example, suppose you are given the demand function $Q_d = 6-2P$, where Q is in thousand kilograms (kg) of good Zeta per week, and P is price in $; you are told that following a successful advertising campaign for Zeta, there is increase in demand of 2000 kg per week, and you are asked to plot the new demand curve. You must find the new demand function before you can use your GDC. In the new demand function, the Q intercept, which initially was 6, increases by 2 to become 8. Therefore, the new function is:

$$Q_d = 8 - 2P$$

If you put this function into the calculator and ask it to graph the inverse function, you will be shown the new demand curve.

However, *you should ask yourself whether it is worth your time to do this*. If you have already plotted the curve for $Q_d = 6-2P$, and you have found the new Q intercept, which is 8, all you have to do is draw a new demand curve parallel to the initial one, beginning at the Q intercept of $Q = 8$. This is so simple to do, that the use of a GDC would not be a good use of your time.

Changes in the slope

The slope was defined as the change in the dependent variable (Q) divided by the change in the independent variable (P) between any two points on a linear curve, and is the coefficient of P in a demand function or a supply function. Given a demand function such as $Q_d = 6-2P$, suppose you are told that the slope changes to –3. Your GDC cannot be used unless you first find the new function; to do this simply replace –2 by –3, thus getting $Q_d = 6-3P$.

If you were asked to graph the new curve that results after a change in the slope, you could put the new function into the calculator and ask for a graph of the inverse function.

If you were asked whether the new curve will be steeper or flatter than the initial curve (without graphing), you could graph them anyway, and compare the inverse functions of the initial and final demand functions with respect to their steepness; in other words, compare the steepness of the inverse function of $Q_d = 6-2P$ with the steepness of the inverse function of $Q_d = 6-3P$. Note, though that this may be tricky, because when you view both graphs at the same time on your screen, it may be confusing to determine which curve is which.

Therefore, *it may not worth your time to use your GDC for this* purpose. To figure out whether the new curve will be steeper or flatter, all you have to do is remember the simple rule: *the greater the absolute value of the slope, the flatter the curve*. Since –3 has a greater absolute value than –2, you can conclude (without the help of a GDC) that the new curve will be flatter.

Steps for using a GDC to find inverse demand and supply functions and equilibrium price and quantity

The process of graphing inverse functions on a GDC differs according to the type of GDC. The steps for Casio and Texas Instruments (TI) GDCs are explained below. In both cases we will use the same demand and supply functions used above.

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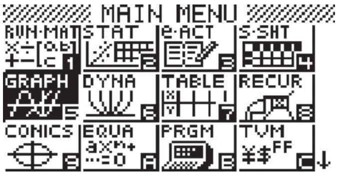
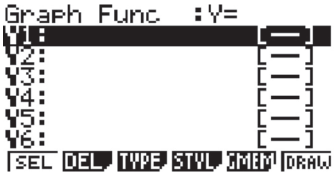
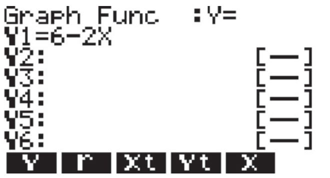
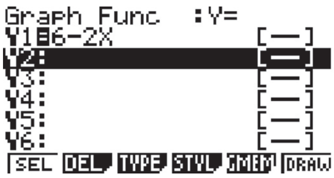
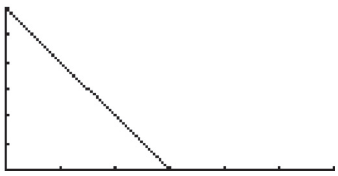
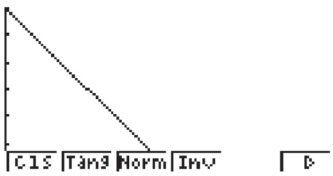
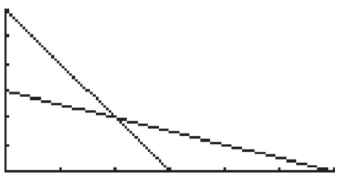
Casio

How to plot an inverse function on a Casio

	1 From the main menu, use the arrow keys to mark the window **GRAPH**.
	2 Press **EXE**.
	3 Type in the function $y = 6 - 2x$. (The variable x is below the red **ALPHA** button.)
	4 Press **EXE**.
	5 Press **F6**. Note that this function corresponds to Figure 24(b), i.e. $Q = 6 - 2s$ *with Q plotted on the vertical axis*.
	6 Press **F4**.
	7 Press **F4** again. (This asks the calculator to draw the inverse function.) You will get the graph of the inverse function, i.e. $Q = 6 - 2x$ *with Q plotted on the horizontal axis*, along with the original function. Note that the inverse function corresponds to Figure 24(a), which is the correct graph of the demand function.

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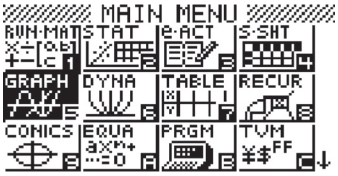
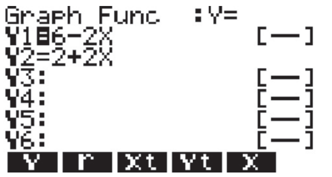
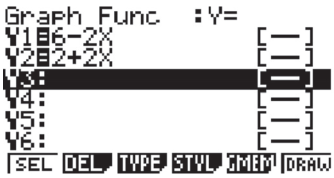
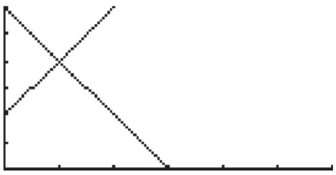
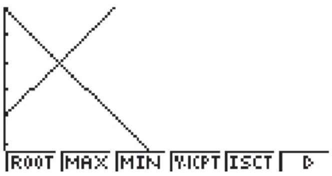
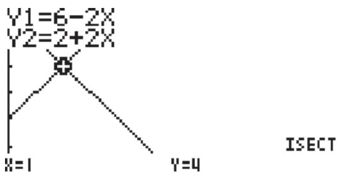
HL How to find the point of intersection of two straight lines (a demand function and a supply function) on a Casio HL

	1 From the main menu, use the arrow keys to mark the window **GRAPH**.
	2 Press **EXE**.
	3 Type in the function $y = 6 - 2x$. Press **EXE** to go to the second line and then type the next function, $y = 2 + 2x$.
	4 Press **EXE**.
	5 Press **F6** to plot both functions. Note that this corresponds to Figure 26(b).
	6 Press the **SHIFT** key and then the **F5** (also known as G-Solv) key to display the following window.
	7 Press the **F5** key (for intersection). This gives the values of x and y at the point of intersection. However, you must remember that $x = 1$ corresponds to $P = 1$ and $y = 4$ to $Q = 4$; therefore, while the point of intersection is correct, the curves are incorrect, *because they plot Q on the vertical axis*.

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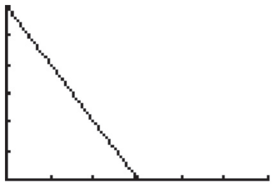
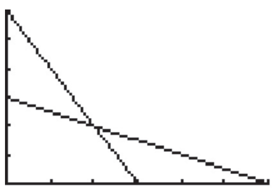
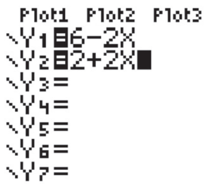
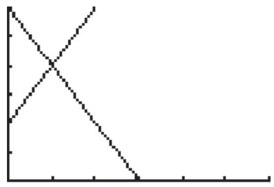
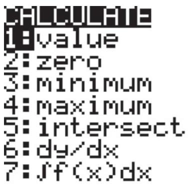
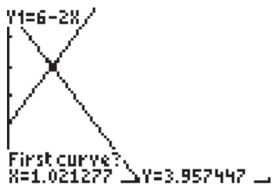
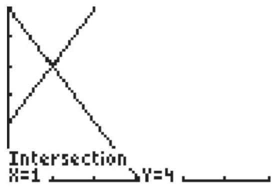
Texas Instruments

How to plot an inverse function on a TI

Plot1 Plot2 Plot3 \Y1=6-2X \Y2= \Y3= \Y4= \Y5= \Y6= \Y7=	1 Press the **Y =** key and type in the function $y = 6 - 2x$. (You will find the symbol x next to the green **ALPHA** button.)
	2 Press the **GRAPH** key to plot this function. Note that this function corresponds to Figure 24(b), i.e. $Q = 6 - 2x$ *with Q plotted on the vertical axis*.
DRAW POINTS STO 1:ClrDraw 2:Line(3:Horizontal 4:Vertical 5:Tangent(6:DrawF 7↓Shade(	3 Press the **2ND** key and then the **DRAW** key to display the following window.
DrawInv	4 Scroll down to number **8** or just press **8**. (This asks the calculator to draw the inverse function.)
VARS Y-VARS 1:Window… 2:Zoom… 3:GDB… 4:Picture… 5:Statistics… 6:Table… 7:String…	5 Press the **VARS** key.
VARS Y-VARS 1:Function… 2:Parametric… 3:Polar… 4:On/Off…	6 Press the right arrow to mark Y-VARS and then press **ENTER**.
FUNCTION 1:Y1 2:Y2 3:Y3 4:Y4 5:Y5 6:Y6 7↓Y7	7 This window is displayed. Press **ENTER**.
DrawInv Y1	8 This window is displayed.
	9 Press **ENTER** again. You will see the graph of the inverse function, i.e. $Q = 6 - 2x$ *with Q plotted on the horizontal axis*, along with the original function. Note that the inverse function corresponds to Figure 24(a), which is the correct graph of the demand function.

HL

How to find the point of intersection of two straight lines (a demand function and a supply function) on a TI

	1 Press the **Y =** key and type in the function $y = 6 - 2x$. Press **ENTER** to go to the second line and then type the next function, $y = 2 + 2x$.
	2 Press **GRAPH** to plot both functions. Note that this corresponds to Figure 26(b).
	3 Press the **2ND** key and then the **CALC** key to display the following window.
	4 Press the number **5** (for intersection).
	5 Press **ENTER** 3 times. This gives the values of x and y at the point of intersection. However, you must remember that $x = 1$ corresponds to $P = 1$ and $y = 4$ to $Q = 4$; therefore, while the point of intersection is correct, the curves are incorrect, *because they plot Q on the vertical axis.*

It may be noted that on both the Casio and the TI, it is also possible to use the simultaneous equation facility to find the point of intersection of two lines.

Concluding comments

The main use of a GDC for higher level paper 3 is as an aid to graphing demand and supply curves. For this purpose, you must learn how to find graphs of inverse functions on your calculator, where P is expressed as a function of Q. Beyond that, you must understand the meaning of the parameters of the demand and supply functions (the Q intercept and the slope) and how to find a new function given a change in any of these, as the GDC can be of no use to you in this respect. If you learn how to plot the curves of demand and supply functions quickly and accurately, it is most likely that a GDC will not be of much use to you, except for simple arithmetic calculations, and possibly as a check for the accuracy of your work.

Test your understanding 14

Answer the questions below for each of the following demand and supply equations.

(i) $Q_d = 10 - 5P$; $Q_s = 2 + 3P$

(ii) $Q_d = 45 - 5P$; $Q_s = -30 + 20P$

(iii) $Q_d = 60 - 20P$; $Q_s = 20 + 20P$

1 Graph the demand and supply equations, and find the equilibrium price and quantity on your graph (a) without the use of a GDC, and (b) using a GDC to find the inverse functions. (c) Compare your results to make sure they agree.

2 Compare the two methods and determine which of the two you find more convenient.

3 Find the equilibrium price and quantity mathematically, and compare with the results of your graph to make sure they agree.

4 Find the equilibrium price and quantity using your GDC, and compare with your results in your graph.